

COMMUTANT LIFTING, INTERPOLATION, AND PERTURBATIONS ON THE POLYDISC

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ABSTRACT. The fundamental theorem on commutant lifting due to Sarason does not carry over to the setting of the polydisc. This paper presents two classifications of commutant lifting in several variables. The first classification links the lifting problem to the contractivity of certain linear functionals. The second one transforms it into nonnegative real numbers. We also solve the Nevanlinna-Pick interpolation problem for bounded analytic functions on the polydisc. Along the way, we solve a perturbation problem for bounded analytic functions. Commutant lifting and interpolation on the polydisc solve two well-known problems in Hilbert function space theory.

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1. INTRODUCTION

Sarason's commutant lifting theorem [60] is fundamental, with significant applications to virtually every aspect of Hilbert function space theory. One of them is the Nevanlinna-Pick interpolation theorem on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which we will quickly review before moving on to the lifting theorem. Given m distinct points $\mathcal{Z} = \{z_1, \dots, z_m\} \subset \mathbb{D}$ (interpolation nodes) and m scalars $\mathcal{W} = \{w_1, \dots, w_m\} \subset \mathbb{D}$

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(target data), there exists an analytic function $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ (interpolating function) such that

$$\sup_{z \in \mathbb{D}} |\varphi(z)| \leq 1$$

and

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if the $m \times m$ *Pick matrix* $\mathfrak{P}_{\mathcal{Z}, \mathcal{W}}$ is positive semi-definite, where

$$\mathfrak{P}_{\mathcal{Z}, \mathcal{W}} := \left(\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right)_{i,j=1}^m.$$

This was proved by G. Pick [54] more than a century ago. R. Nevanlinna [48] independently solved the same problem at a very similar time. The methods of Pick and Nevanlinna are different, interesting on their own, and still relevant. For instance, Pick focused on interpolation on the upper half-plane, whereas the Schur algorithm (see I. Schur [62, 63]) served as the driving force behind Nevanlinna's strategy [36, 50].

After four decades of Pick's paper, D. Sarason [60] provided a solid Hilbert function space theoretical foundation for Nevanlinna and Pick's analytic and algebraic methods for the solution of the interpolation problem. Sarason's elegant result, known as the *commutant lifting theorem*, represents the commutant of model operators in terms of nicer operators (say Toeplitz operators) without changing the norms. To be more specific, let us recall that the Hardy space $H^2(\mathbb{T}^n)$ on the polydisc \mathbb{D}^n is the space of all analytic functions f on \mathbb{D}^n such that

$$\|f\|_2 := \left(\sup_{0 < r < 1} \int_{\mathbb{T}^n} |f(rz)|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\mu$ denotes the normalized Lebesgue measure on \mathbb{T}^n , $z = (z_1, \dots, z_n)$, and $rz = (rz_1, \dots, rz_n)$. Note that $\mathbb{T}^n = \partial\mathbb{D}^n$ is the Šilov boundary of \mathbb{D}^n . We denote the von Neumann algebra of essentially bounded Lebesgue measurable functions on \mathbb{T}^n by $L^\infty(\mathbb{T}^n)$. The analytic counterpart of $L^\infty(\mathbb{T}^n)$ is $H^\infty(\mathbb{D}^n)$, where

$$H^\infty(\mathbb{D}^n) = \{\varphi \in \mathcal{O}(\mathbb{D}^n) : \|\varphi\|_\infty := \sup_{z \in \mathbb{D}^n} |\varphi(z)| < \infty\},$$

the Banach algebra of bounded analytic functions on \mathbb{D}^n equipped with the supremum norm $\|\cdot\|_\infty$. The space $H^\infty(\mathbb{D}^n)$ is known to be one of the most sophisticated commutative Banach algebras. Even at the level of a single variable, many questions in analytic function theory and harmonic analysis remain to be settled (cf. [10, 17] and the references therein). Clearly, $H^\infty(\mathbb{D}^n) \subseteq H^2(\mathbb{T}^n)$. The class of functions of interest is the closed unit ball of $H^\infty(\mathbb{D}^n)$:

$$\mathcal{S}(\mathbb{D}^n) = \{\varphi \in H^\infty(\mathbb{D}^n) : \|\varphi\|_\infty \leq 1\}.$$

The members of $\mathcal{S}(\mathbb{D}^n)$ are known as *Schur functions*. Given $\varphi \in H^\infty(\mathbb{D}^n)$, the *analytic Toeplitz operator* T_φ on $H^2(\mathbb{T}^n)$ is defined by

$$T_\varphi f = \varphi f,$$

for all $f \in H^2(\mathbb{T}^n)$. In particular, for $\varphi = z_i$, we get T_{z_i} the multiplication operator by coordinate function z_i on $H^2(\mathbb{T}^n)$, $i = 1, \dots, n$. The following equality describes how the commutant of $\{T_{z_i}\}_{i=1}^n$ connects the Banach algebra $H^\infty(\mathbb{D}^n)$ to $\mathcal{B}(H^2(\mathbb{T}^n))$:

$$\{T_{z_1}, \dots, T_{z_n}\}' = \{T_\varphi : \varphi \in H^\infty(\mathbb{D}^n)\}.$$

Moreover, it follows that [28]

$$\|T_\varphi\| = \|\varphi\|_\infty \quad (\varphi \in H^\infty(\mathbb{D}^n)).$$

We now return to the classical case where $n = 1$. Let \mathcal{Q} be a T_z^* -invariant closed subspace of $H^2(\mathbb{T})$, and let X be a bounded linear operator on \mathcal{Q} (in short, $X \in \mathcal{B}(\mathcal{Q})$). Sarason's commutant lifting theorem states the following: Suppose X commutes with the *model operator* $P_{\mathcal{Q}}T_z|_{\mathcal{Q}} \in \mathcal{B}(\mathcal{Q})$, that is

$$X(P_{\mathcal{Q}}T_z|_{\mathcal{Q}}) = (P_{\mathcal{Q}}T_z|_{\mathcal{Q}})X.$$

Then there exists $\varphi \in H^\infty(\mathbb{D})$ such that

$$X = P_{\mathcal{Q}}T_\varphi|_{\mathcal{Q}},$$

and

$$\|X\| = \|\varphi\|_\infty.$$

Here (and in what follows) $P_{\mathcal{Q}}$ denotes the orthogonal projection from $H^2(\mathbb{T})$ onto \mathcal{Q} . In other words, along with $\|X\| = \|T_\varphi\|$, the following diagram commutes:

$$\begin{array}{ccc} H^2(\mathbb{T}) & \xrightarrow{T_\varphi} & H^2(\mathbb{T}) \\ i_{\mathcal{Q}} \uparrow & & \downarrow P_{\mathcal{Q}} \\ \mathcal{Q} & \xrightarrow{X} & \mathcal{Q} \end{array}$$

where $i_{\mathcal{Q}} : \mathcal{Q} \hookrightarrow H^2(\mathbb{T})$ denotes the inclusion map. The Nevanlinna-Pick interpolation theorem then easily follows from this applied to zero-based finite-dimensional T_z^* -invariant subspaces of $H^2(\mathbb{T})$ (cf. Subsection 10.2). The most important aspect of Sarason's lifting theorem, however, is the lifting of the commutant of model operators to the commutant of T_z keeping the norms the same.

We remind the reader that Sarason's commutant lifting theorem has a stellar reputation in its application to the classical operator and function theoretic results like the Carathéodory-Fejér interpolation problem, Nehari interpolation problem, von Neumann inequality, isometric dilations, and the Ando dilation, just to name a few. The expanded list easily includes control theory and electrical engineering [34, 40]. When dealing with several variables, however, each analogue question poses a unique set of challenges and frequently offers less opportunity for a comprehensive theory (however, see [7, 12, 13, 27, 29, 37]). In fact, it is known that Sarason's commutant lifting theorem does not hold true in general in several variables (see Section 3). Understanding the obstacle of commutant lifting in several variables is thus one of the most important problems in Hilbert function space theory.

In this paper, we solve the commutant lifting problem on $H^2(\mathbb{T}^n)$, $n \geq 1$. That is, given a closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ that is invariant under $T_{z_i}^*$, $i = 1, \dots, n$, we classify contractions $X \in \mathcal{B}(\mathcal{Q})$ satisfying the condition that

$$X(P_{\mathcal{Q}}T_{z_i}|_{\mathcal{Q}}) = (P_{\mathcal{Q}}T_{z_i}|_{\mathcal{Q}})X \quad (i = 1, \dots, n),$$

so that the following diagram commutes

$$\begin{array}{ccc}
 H^2(\mathbb{T}^n) & \xrightarrow{T_\varphi} & H^2(\mathbb{T}^n) \\
 i_{\mathcal{Q}} \uparrow & & \downarrow P_{\mathcal{Q}} \\
 \mathcal{Q} & \xrightarrow{X} & \mathcal{Q}
 \end{array}$$

for some $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Several attempts have been made to solve this problem, but they appear to be quite abstract and only applicable to a smaller class of operators (or functions). The most notable one is perhaps the work of Ball, Li, Timotin, and Trent [12]. The class of functions considered in [12] is the so-called Schur-Agler class functions. This class is significantly smaller than even the polydisc algebra when $n > 2$, and it is the same as the Schur class when $n = 2$. Even in the $n = 2$ case, however, the existing results are abstract. In the context of interpolation for the $n = 2$ case, we refer the reader to the seminal papers by Agler [5, 6] (also, see the discussion following Theorem 1.5).

Our approach and solution to the commutant lifting problem are both concrete and function-theoretic. As part of the application, we solve the interpolation problem for Schur functions on \mathbb{D}^n . Moreover, in the context of Schur functions on \mathbb{D}^n , we also solve a perturbation problem. Like our commutant lifting theorem, all results are concrete and quantify the complexity of the problem by nonnegative real numbers.

Now we provide a more thorough summary of this paper's key contribution. Unless otherwise specified, we will always assume that $n \geq 1$ is a natural number. Given a Hilbert space \mathcal{H} , set

$$\mathcal{B}_1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq 1\}.$$

Given a nonempty subset $S \subseteq H^2(\mathbb{T}^n)$, we define the conjugate space S^{conj} as

$$S^{conj} = \{\bar{f} : f \in S\}.$$

Let $\mathcal{S} \subseteq H^2(\mathbb{T}^n)$ be a closed subspace. We say that \mathcal{S} is a *shift invariant subspace* (or submodule) if

$$z_i \mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \dots, n$. We say that \mathcal{S} is a *backward shift invariant subspace* (or quotient module) if \mathcal{S}^\perp is a shift invariant subspace, or equivalently,

$$T_{z_i}^* \mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \dots, n$. Given a backward shift invariant subspace $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$, we define the *model operator* S_{z_i} , for each $i = 1, \dots, n$, by

$$S_{z_i} = P_{\mathcal{Q}} T_{z_i}|_{\mathcal{Q}}.$$

Now we define lifting on backward shift invariant subspaces (also see Definition 2.2).

Definition 1.1. Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a backward shift invariant subspace, $X \in \mathcal{B}_1(\mathcal{Q})$, and suppose $X S_{z_i} = S_{z_i} X$ for all $i = 1, \dots, n$. If there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$X = P_{\mathcal{Q}} T_\varphi|_{\mathcal{Q}},$$

then X is said to have a lift, or to be liftable.

Before we get into the main contribution of this paper, we need to familiarise ourselves with a few additional concepts. First, we define the closed subspace of “mixed functions” of $L^2(\mathbb{T}^n)$ as

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{\text{conj}} + H^2(\mathbb{T}^n)).$$

This space has a significant role to perform in the entire paper. It is crucial to observe that $\mathcal{M}_n \cap H^2(\mathbb{T}^n) = \{0\}$, and

$$\mathcal{M}_1 = \{0\}.$$

Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a backward shift invariant subspace. Set

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{\text{conj}} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)), \quad (1.1)$$

where

$$H_0^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) \ominus \{1\},$$

the closed subspace of $H^2(\mathbb{T}^n)$ of functions vanishing at the origin. Note that $\dot{+}$ signifies the skew sum of Banach spaces. In what follows, we treat $\mathcal{M}_{\mathcal{Q}}$ as a subspace of the classical Banach space $L^1(\mathbb{T}^n)$, and denote it by $(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1)$. In other words

$$(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \subset (L^1(\mathbb{T}^n), \|\cdot\|_1).$$

Let $X \in \mathcal{B}(\mathcal{Q})$, and suppose

$$\psi = X(P_{\mathcal{Q}}1). \quad (1.2)$$

Define a functional $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \rightarrow \mathbb{C}$ by

$$X_{\mathcal{Q}}f = \int_{\mathbb{T}^n} \psi f \, d\mu \quad (f \in \mathcal{M}_{\mathcal{Q}}).$$

Recall that $d\mu$ is the normalized Lebesgue measure on \mathbb{T}^n . Finally, set

$$\tilde{\mathcal{M}}_{\mathcal{Q},X} = (\mathcal{Q}^{\text{conj}} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)),$$

and again treat it as a subspace of $L^1(\mathbb{T}^n)$:

$$(\tilde{\mathcal{M}}_{\mathcal{Q},X}, \|\cdot\|_1) \subset (L^1(\mathbb{T}^n), \|\cdot\|_1).$$

Now that we have these notations, we can say how the lifting of commutants in higher dimensions is classified (see Theorem 4.4):

Theorem 1.2. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a backward shift invariant subspace and let $X \in \mathcal{B}(\mathcal{Q})$ be a contraction. Suppose $XS_{z_i} = S_{z_i}X$ for all $i = 1, \dots, n$. The following conditions are equivalent:*

- (1) X admits a lift.
- (2) $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \rightarrow \mathbb{C}$ is a contractive functional, where

$$X_{\mathcal{Q}}f = \int_{\mathbb{T}^n} \psi f \, d\mu \quad (f \in \mathcal{M}_{\mathcal{Q}}).$$

- (3) $\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q},X}\right) \geq 1$.

This solves the long-standing commutant lifting problem for $H^2(\mathbb{T}^n)$, $n > 1$. We believe that the technique used to prove our lifting theorem is interesting on its own. In fact, within the framework of the application to a different flavor, we solve a perturbation problem of independent interest: Given a nonzero function $f \in H^2(\mathbb{T}^n)$, does there exist $g \in H^2(\mathbb{T}^n)$ such that

$$f + g \in \mathcal{S}(\mathbb{D}^n)?$$

Of course, to avoid triviality (that $g = -f$, for instance), we must assume that $g \in \{f\}^\perp$. Set

$$\mathcal{L}_n = \mathcal{M}_n \oplus H_0^2(\mathbb{T}^n),$$

and treat it as a subspace of $L^1(\mathbb{T}^n)$. Theorem 5.1 presents a complete solution to this problem:

Theorem 1.3. *Let $f \in H^2(\mathbb{T}^n)$ be a nonzero function. Then there exists $g \in \{f\}^\perp$ such that*

$$f + g \in \mathcal{S}(\mathbb{D}^n),$$

if and only if

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{f}}{\|f\|_2^2}, \mathcal{L}_n\right) \geq 1.$$

Now we will explain the solution to the interpolation problem, which also resolves the long-standing question on interpolation with Schur functions as interpolating functions on \mathbb{D}^n , $n > 1$. We will start by laying the groundwork. Recall that $H^2(\mathbb{T}^n)$ is a reproducing kernel Hilbert space corresponding to the *Szegő kernel* $\mathbb{S} : \mathbb{D}^n \times \mathbb{D}^n \rightarrow \mathbb{C}$, where

$$\mathbb{S}(z, w) = \prod_{i=1}^n \frac{1}{1 - z_i \bar{w}_i} \quad (z, w \in \mathbb{D}^n).$$

For each $w \in \mathbb{D}^n$, define $\mathbb{S}(\cdot, w) : \mathbb{D}^n \rightarrow \mathbb{C}$ by $(\mathbb{S}(\cdot, w))(z) = \mathbb{S}(z, w)$ for all $z \in \mathbb{D}^n$. In view of the standard reproducing kernel property, it follows that $\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}^n\} \subseteq H^2(\mathbb{T}^n)$ is a set of linearly independent functions, and

$$\mathbb{S}(z, w) = \langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \rangle_{H^2(\mathbb{T}^n)},$$

for all $z, w \in \mathbb{D}^n$. Given a set of distinct points $\mathcal{Z} = \{z_1, \dots, z_m\} \subset \mathbb{D}^n$, we define an m -dimensional subspace of $H^2(\mathbb{T}^n)$ as

$$\mathcal{Q}_{\mathcal{Z}} = \text{span}\{\mathbb{S}(\cdot, z_j) : j = 1, \dots, m\}.$$

It follows that $\mathcal{Q}_{\mathcal{Z}}$ is a backward shift invariant subspace of $H^2(\mathbb{T}^n)$. Define

$$\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}} = \mathcal{Q}_{\mathcal{Z}}^{\text{conj}} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

In addition, given a set of scalars $\{w_i\}_{i=1}^m \subset \mathbb{D}$, define $X_{\mathcal{Z}, \mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by

$$X_{\mathcal{Z}, \mathcal{W}}^* \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j) \quad (j = 1, \dots, m).$$

The fact that $X_{\mathcal{Z}, \mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$ is a natural operator and that it meets the crucial condition that $X_{\mathcal{Z}, \mathcal{W}} S_{z_i} = S_{z_i} X_{\mathcal{Z}, \mathcal{W}}$, $i = 1, \dots, m$, is noteworthy (see Lemma 6.3).

Here is a summary of our main interpolation results (see Theorem 6.6 and Theorem 7.1):

Theorem 1.4. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. The following conditions are equivalent:*

- (1) *There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, \dots, m$.*
- (2) *$\mathfrak{I}_{\mathcal{Z}, \mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \rightarrow \mathbb{C}$ is a contraction, where*

$$\mathfrak{I}_{\mathcal{Z}, \mathcal{W}} f = \int_{\mathbb{T}^n} \psi_{\mathcal{Z}, \mathcal{W}} f \, d\mu,$$

for all $f \in \mathcal{M}_{\mathcal{Q}_Z}$, and

$$\psi_{Z,W} = \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i),$$

and the scalar coefficients $\{c_i\}_{i=1}^m$ are given by

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \mathbb{S}(z_1, z_1) & \mathbb{S}(z_1, z_2) & \cdots & \mathbb{S}(z_1, z_m) \\ \mathbb{S}(z_2, z_1) & \mathbb{S}(z_2, z_2) & \cdots & \mathbb{S}(z_2, z_m) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{S}(z_m, z_1) & \mathbb{S}(z_m, z_2) & \cdots & \mathbb{S}(z_m, z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

(3) Let $\psi := X_{Z,W}(P_{\mathcal{Q}_Z} 1)$, and suppose

$$\tilde{\mathcal{M}}_{Z,W} := (\mathcal{Q}_Z^{\text{conj}} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Then

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_Z}\right) \geq 1.$$

Note that the matrix in part (2) of the above theorem is the inverse of the Gram matrix

$$\left(\mathbb{S}(z_i, z_j)\right)_{i,j=1}^m,$$

corresponding to the m Szegő kernel functions $\{\mathbb{S}(\cdot, z_i)\}_{i=1}^m$. Also, observe that part (3) provides a useful quantitative criterion to check interpolation on the polydisc. Indeed, as shown in Theorem 7.4, the quantitative criterion yields examples of interpolation on \mathbb{D}^n , $n \geq 2$. Notable is the fact that interpolating functions in this case are polynomials.

It is noteworthy that the answer to natural questions, as in Theorems 1.2, 1.3, and 1.4, has a connection to the set of nonnegative real numbers. This is a common and classical occurrence. The classical Nehari theorem [49], for example, establishes a direct link with such a distance function. Another instance is the celebrated Adamyan-Arov-Krein formulae [2, 3, 4]. We will comment some more at the end of Section 5.

In Theorem 9.2, we recover Sarason's lifting theorem as an application to Theorem 1.2, resulting in yet another proof of the classical lifting theorem:

Theorem 1.5. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T})$ be a backward shift invariant subspace, and let $X \in \mathcal{B}_1(\mathcal{Q})$. If*

$$XS_z = S_z X,$$

then X is liftable.

In the proof of the above theorem, $\mathcal{M}_{\mathcal{Q}}$ (defined as in (1.1)) admits a more compact form, namely

$$\mathcal{M}_{\mathcal{Q}} = \overline{\varphi}(zH^2(\mathbb{T})),$$

where $\varphi \in H^\infty(\mathbb{D})$ is an inner function (that is, $|\varphi| = 1$ on \mathbb{T} a.e.) and $\mathcal{Q} = (\varphi H^2(\mathbb{T}))^\perp$. Moreover, we employ all the standard one variable types of machinery like the Beurling theorem, inner-outer factorizations [15, 36], etc. On the one hand, this is to be expected given that Sarason uses similar tools for his lifting theorem. This, on the other hand, explains both the challenges associated with the commutant lifting theorem and the potential for extensions of relevant function theoretic results on the polydisc.

Here are some additional facts and thoughts regarding the commutant lifting and interpolation problems, as well as the context of our approach. In 1968, Sz.-Nagy and Foiaş

[44] generalized the Sarason lifting theorem to vector-valued Hardy spaces. In subsequent papers, many researchers presented a variety of alternative proofs of independent interest (cf. [9, 30, 59]). However, the dilation theory (pioneered by Halmos [38] and advanced by Sz.-Nagy [43]) is the primary technique employed in all of these papers which is powerful enough to negate the heavy use of function theoretic tools. For different versions of the commutant lifting theorem and its applications, we refer to Bercovici, Foiaş and Tannenbaum [14], and the monographs by Nikolski [52], Sz.-Nagy and Foiaş [66], and Foiaş and Frazho [35] (also see Nikolski and Volberg [53] and Seip [64]).

In several variables, the earlier approach to the lifting theorem also appears to be dilation theoretic or under the assumption of von Neumann inequality, where dilation theory and von Neumann inequality for commuting contractions are complex subjects in and of themselves.

On the other hand, if the solution to the interpolation problem on \mathbb{D}^n , $n \geq 1$, is sought in terms of the Pick matrix's positive semi-definiteness, then the interpolation problem becomes equivalent to the commutant lifting theorem on finite-dimensional zero-based subspaces (cf. Proposition 10.5). Consequently, in one variable, thanks to Sarason, the commutant lifting property, the Pick positivity, and the solution to the interpolation problem appear to be inextricably linked. In higher variables, however, because the commutant lifting property is rather erratic (cf. Section 3), it is perhaps necessary to disencumber the positivity of the Pick matrix from the interpolation problem. In some ways, these observations seek a different perspective on the several variables interpolation problem, one that is not as similar to the classical case of positivity of the Pick matrix (nor even positivity of a family of Pick matrices as in [1, 19, 29, 39]). As a consequence, we approach the problem from a completely different angle: more along the function theoretic path pioneered by Sarason. The difficulty here, of course, is dealing with the sensitivity of several complex variables as well as the lack of all standard one variable tools.

Finally, a few words about this paper's methodology. We heavily use the duality of classical Banach spaces, namely

$$(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n).$$

Other common tools used in this paper include the classical Hahn-Banach theorem, the geometry of Banach spaces, and the Hilbert function space theory.

The remainder of the paper is structured as follows. Section 2 introduces some preliminary concepts. Section 3 outlines explicit examples of non-liftable maps. Section 4 presents classifications of commutant lifting. Section 5 solves the perturbation problem of $H^2(\mathbb{T}^n)$ -functions in terms of Schur functions. Section 6 presents the first classification of the interpolation on \mathbb{D}^n . A quantitative classification for interpolation is presented in Section 7. In the same section, by using the quantitative classification, we provide examples of interpolation on \mathbb{D}^n , $n \geq 2$. The commutant lifting theorem on \mathbb{D}^n is tested in Section 8 with some concrete examples. As an application to our main commutant lifting theorem, Section 9 provides new proof for the classical lifting theorem. In Section 10 we make some general observations such as the Carathéodory-Fejér interpolation problem, weak interpolation, and decomposing a polynomial as a sum of bounded analytic functions. Section 11 concludes with some closing remarks and thoughts on some other known results.

The paper contains an abundance of examples and counterexamples, as well as numerous auxiliary results of independent interest in both one and several variables. This paper is nearly self-contained.

2. PRELIMINARIES

In this section, we will introduce some necessary Hilbert function space theoretic preliminaries. These include Hardy space, submodules, quotient modules, and a formal definition of lifting. We begin by looking at the Hardy space. We again remind the reader that throughout the paper, n will denote a natural number, and (unless otherwise stated) we always assume that $n \geq 1$.

We denote as usual by $L^2(\mathbb{T}^n)$ the space of square-integrable functions on \mathbb{T}^n . Recall that \mathbb{T}^n is the Šilov boundary of \mathbb{D}^n . The *Hardy space* $H^2(\mathbb{T}^n)$ is the closed subspace of $L^2(\mathbb{T}^n)$ consisting of those functions whose Fourier coefficients vanish off \mathbb{Z}_+^n . More specifically, consider $f \in L^2(\mathbb{T}^n)$ with Fourier series representation

$$f = \sum_{k \in \mathbb{Z}^n} a_k z^k \quad (z \in \mathbb{T}^n),$$

where $z^k = z_1^{k_1} \cdots z_n^{k_n}$ for all $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$. Then $f \in H^2(\mathbb{T}^n)$ if and only if $a_k = 0$ whenever at least one of the k_j , $j = 1, \dots, n$, in $k = (k_1, \dots, k_n)$ is negative. The usage of radial limits is another neat way to represent the Hardy space (see Rudin [56]). In other words, we will identify $H^2(\mathbb{T}^n)$ with $H^2(\mathbb{D}^n)$, the Hilbert space analytic functions $f \in \mathcal{O}(\mathbb{D}^n)$ such that

$$\|f\|_2 := \left(\sup_{0 < r < 1} \int_{\mathbb{T}^n} |f(rz)|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty, \quad (2.1)$$

where $d\mu$ denotes the normalized Lebesgue measure on \mathbb{T}^n , and $rz = (rz_1, \dots, rz_n)$. The identification is canonical, that is, given $f \in H^2(\mathbb{D}^n)$, the radial limit

$$\tilde{f}(z) := \lim_{r \rightarrow 1^-} f(rz),$$

exists for almost every $z \in \mathbb{T}^n$, and $\tilde{f} \in H^2(\mathbb{T}^n)$, and vice-versa. In what follows (and unless otherwise stated) we will not distinguish between $f \in \mathcal{O}(\mathbb{D}^n)$ satisfying (2.1) and its radial limit representation $\tilde{f} \in H^2(\mathbb{T}^n)$. Therefore, we will not distinguish between $H^2(\mathbb{T}^n)$ and $H^2(\mathbb{D}^n)$ and will use the same notation $H^2(\mathbb{T}^n)$ for both.

It is frequently useful to represent $H^2(\mathbb{T}^n)$ as the Hilbert space of square-summable analytic functions on \mathbb{D}^n , that is

$$H^2(\mathbb{T}^n) = \left\{ \sum_{k \in \mathbb{Z}_+^n} a_k z^k \in \mathcal{O}(\mathbb{D}^n) : \sum_{k \in \mathbb{Z}_+^n} |a_k|^2 < \infty \right\}.$$

The Hardy space $H^2(\mathbb{T}^n)$ is equipped with the tuple of multiplication operators by coordinate functions $\{z_1, \dots, z_n\}$, which we denote by $(T_{z_1}, \dots, T_{z_n})$. Therefore, by definition, we have

$$(T_{z_i} f)(w) = w_i f(w),$$

for all $f \in H^2(\mathbb{T}^n)$, $w \in \mathbb{D}^n$, and $i = 1, \dots, n$. It is easy to see that $(T_{z_1}, \dots, T_{z_n})$ is an n -tuple of commuting isometries, that is

$$T_{z_i}^* T_{z_i} = I_{H^2(\mathbb{T}^n)}, \text{ and } T_{z_i} T_{z_j} = T_{z_j} T_{z_i},$$

for all $i, j = 1, \dots, n$. We will also need to use the doubly commutativity property

$$T_{z_i}^* T_{z_j} = T_{z_j} T_{z_i}^* \quad (i \neq j).$$

From the analytic function space perspective, recall that $H^2(\mathbb{T}^n)$ is a reproducing kernel Hilbert space corresponding to the *Szegő kernel* \mathbb{S} on \mathbb{D}^n , where

$$\mathbb{S}(z, w) = \prod_{i=1}^n \frac{1}{1 - z_i \bar{w}_i} \quad (z, w \in \mathbb{D}^n).$$

For each $w \in \mathbb{D}^n$, the *kernel function* $\mathbb{S}(\cdot, w) : \mathbb{D}^n \rightarrow \mathbb{C}$ defined by

$$(\mathbb{S}(\cdot, w))(z) = \mathbb{S}(z, w) \quad (z \in \mathbb{D}^n),$$

generates the joint eigenspace of the backward shifts, that is

$$\bigcap_{i=1}^n \ker(T_{z_i} - w_i I_{H^2(\mathbb{T}^n)})^* = \mathbb{C}\mathbb{S}(\cdot, w). \quad (2.2)$$

The above equality essentially follows from the fact that

$$T_{z_i}^* \mathbb{S}(\cdot, w) = \bar{w}_i \mathbb{S}(\cdot, w), \quad (2.3)$$

for all $w \in \mathbb{D}^n$ and $i = 1, \dots, n$, and

$$\sum_{k \in \{0,1\}^n} (-1)^{|k|} T_z^k T_z^{*k} = P_{\mathbb{C}},$$

where $P_{\mathbb{C}}$ is the orthogonal projection onto the space of constant functions, and $T_z^k = T_{z_1}^{k_1} \dots T_{z_n}^{k_n}$ for all $k \in \{0,1\}^n \subset \mathbb{Z}_+^n$. Moreover, the set of kernel functions $\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}^n\}$ forms a total set in $H^2(\mathbb{T}^n)$ and satisfy the *reproducing property*

$$f(w) = \left\langle f, \mathbb{S}(\cdot, w) \right\rangle_{H^2(\mathbb{T}^n)}, \quad (2.4)$$

for all $f \in H^2(\mathbb{T}^n)$ and $w \in \mathbb{D}^n$.

Recall from Section 1 that a closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ is called a *quotient module* if $T_{z_i}^* \mathcal{Q} \subseteq \mathcal{Q}$ for all $i = 1, \dots, n$. A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{T}^n)$ is called a *submodule* if $z_i \mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \dots, n$. Equivalently, $\mathcal{S}^\perp \cong H^2(\mathbb{T}^n)/\mathcal{S}$ is a quotient module. In summary, we have the following identifications:

$$\{\text{submodules}\} \longleftrightarrow \{\text{shift invariant subspaces}\},$$

and

$$\{\text{quotient modules}\} \longleftrightarrow \{\text{backward shift invariant subspaces}\}.$$

The classical *Laurent operator* L_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is the bounded linear operator on $L^2(\mathbb{T}^n)$ defined by

$$L_\varphi f = \varphi f,$$

for all $f \in L^2(\mathbb{T}^n)$. The corresponding *Toeplitz operator* is the compression of L_φ to $H^2(\mathbb{T}^n)$, that is

$$T_\varphi f = P_{H^2(\mathbb{T}^n)}(\varphi f),$$

for all $f \in H^2(\mathbb{T}^n)$. As usual, $P_{H^2(\mathbb{T}^n)}$ denotes the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{T}^n)$. Recall that (see [28])

$$\|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{T}^n))} = \|L_\varphi\|_{\mathcal{B}(L^2(\mathbb{T}^n))} = \|\varphi\|_\infty, \quad (2.5)$$

for all $\varphi \in L^\infty(\mathbb{T}^n)$. It is useful to point out that the Toeplitz operator with analytic symbol $\varphi \in H^\infty(\mathbb{D}^n)$ is given by

$$T_\varphi = L_\varphi|_{H^2(\mathbb{T}^n)}.$$

This follows from the general fact that if \mathcal{S} is a submodule of $H^2(\mathbb{T}^n)$, then $\varphi\mathcal{S} \subseteq \mathcal{S}$ for all $\varphi \in H^\infty(\mathbb{D}^n)$. Finally, given a quotient module \mathcal{Q} of $H^2(\mathbb{T}^n)$ and an analytic symbol $\varphi \in H^\infty(\mathbb{D}^n)$, we define the compression operator S_φ on \mathcal{Q} by

$$S_\varphi = P_{\mathcal{Q}}T_\varphi|_{\mathcal{Q}}.$$

In particular, for each $i = 1, \dots, n$, we have the compression of T_{z_i} on \mathcal{Q} as

$$S_{z_i} = P_{\mathcal{Q}}T_{z_i}|_{\mathcal{Q}}.$$

Clearly, $S_\varphi S_{z_i} = S_{z_i} S_\varphi$ for all $i = 1, \dots, n$. From this point of view, we also call that S_φ a module map. In general:

Definition 2.1. Let \mathcal{Q} be a quotient module of $H^2(\mathbb{T}^n)$. An operator $X \in \mathcal{B}(\mathcal{Q})$ is said to be a module map if

$$XS_{z_i} = S_{z_i}X \quad (i = 1, \dots, n).$$

Another common name for module maps is *truncated Toeplitz operators* (even with $L^\infty(\mathbb{T}^n)$ -symbols). See Sarason [57] and also the classic by Brown and Halmos [16].

We conclude this section with the definition of the central concept of this paper. Given a Hilbert space \mathcal{H} , recall again that

$$\mathcal{B}_1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq 1\}.$$

Definition 2.2. Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. If there is a $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$X = S_\varphi,$$

then we say that X has a lift, or X is liftable, or X admits a lift. We also say that φ is a lift of X .

In the case of $n = 1$, Sarason's result states that contractive module maps are always liftable. In the following section, we demonstrate that such a statement is no longer true whenever $n > 1$.

3. HOMOGENEOUS QUOTIENT MODULES

The purpose of this section is to outline explicit and basic examples of non-liftable module maps on quotient modules of $H^2(\mathbb{T}^n)$, $n > 1$. Our quotient modules are as simple as homogenous quotient modules and the module maps are compressions of homogeneous polynomials. We begin with a (probably known) classification of inner polynomials on \mathbb{D}^n . A function $\varphi \in H^\infty(\mathbb{D}^n)$ is called *inner* if φ is unimodular a.e. on \mathbb{T}^n (in the sense of radial limits).

Lemma 3.1. *Let p be a nonzero polynomial in $\mathbb{C}[z_1, \dots, z_n]$. Then p is inner if and only if*

$$p = \text{unimodular constant} \times \text{monomial}.$$

Proof. We assume $n > 1$ because the $n = 1$ case is simpler and follows the same line of proof as the $n > 1$ case. By definition, p is inner if and only if $|p| = 1$ on \mathbb{T}^n . The sufficient part is now trivial. For the reverse direction, assume that p is inner. If p is a constant multiple of a monomial, then passing to the boundary value, the assertion will follow immediately. Therefore, assume that p has more than one term. There exists $N_1 \in \mathbb{N}$ such that

$$p = \sum_{j=0}^{N_1} z_1^j p_j,$$

where $p_j \in \mathbb{C}[z_2, \dots, z_n]$ for all $j = 0, 1, \dots, N_1$, and

$$p_{N_1} \neq 0.$$

Here we are assuming without loss of generality that p has a monomial term with z_1 as a factor (otherwise, we pass to the same but with respect to z_2 and so on). Since p is inner, on \mathbb{T}^n , we have

$$\begin{aligned} 1 &= p\bar{p} \\ &= \bar{z}_1^{N_1} (p_0 \bar{p}_{N_1} \oplus \dots). \end{aligned}$$

This implies

$$p_0 \bar{p}_{N_1} = 0,$$

and hence $p_0 = 0$. Continuing exactly in the same way, we obtain that

$$p = z_1^{N_1} p_{N_1},$$

for some $p_{N_1} \in \mathbb{C}[z_2, \dots, z_n]$. Applying the above recipe to p_{N_1} , we get $p_{N_1} = z_2^{N_2} p_{N_2}$ for some $N_2 \in \mathbb{Z}_+$ and $p_{N_2} \in \mathbb{C}[z_3, \dots, z_n]$. Hence

$$p = z_1^{N_1} z_2^{N_2} p_{N_2}.$$

Therefore, applying this method repeatedly, we finally deduce that p is a unimodular constant multiple of some monomial. \square

Now we turn to the construction of the quotient modules of interest. As is well known and also evident from the definition of the Hardy space, polynomials are dense in $H^2(\mathbb{T}^n)$, that is

$$H^2(\mathbb{T}^n) = \overline{\mathbb{C}[z_1, \dots, z_n]}^{L^2(\mathbb{T}^n)}.$$

Therefore, the standard grading on $\mathbb{C}[z_1, \dots, z_n]$ induces a graded structure on $H^2(\mathbb{T}^n)$. We are essentially going to exploit this simple property in our construction of module maps. For each $t \in \mathbb{Z}_+$, denote by $H_t \subseteq \mathbb{C}[z_1, \dots, z_n]$ the complex vector space of homogeneous polynomials of degree t . We have the vector space direct sum

$$\mathbb{C}[z_1, \dots, z_n] = \bigoplus_{t \in \mathbb{Z}_+} H_t.$$

We consider from now on the finite-dimensional subspace \mathcal{H}_t as a closed subspace of $H^2(\mathbb{T}^n)$. Also, for each $m \in \mathbb{N}$, set

$$\mathcal{Q}_m = \bigoplus_{t=0}^m H_t.$$

Since $T_{z_i}^* \mathcal{Q}_m \subseteq \mathcal{Q}_m$ for all $m \geq 1$, it follows that $\mathcal{Q}_m \subseteq \mathbb{C}[z_1, \dots, z_n]$ is a finite-dimensional quotient module of $H^2(\mathbb{T}^n)$, and $\deg f \leq m$ for all $f \in \mathcal{Q}_m$. Fix $m \in \mathbb{N}$ and fix a homogeneous polynomial of degree m as

$$p = \sum_{|k|=m} a_k z^k \in H_m.$$

Suppose that $\|p\|_2 = 1$. By the definition of the norm on $H^2(\mathbb{T}^n)$, we have

$$\sum_{|k|=m} |a_k|^2 = 1.$$

We aim at investigating the lifting of the module map

$$S_p = P_{\mathcal{Q}_m} T_p|_{\mathcal{Q}_m}.$$

By $S_p f = P_{\mathcal{Q}_m}(pf)$, $f \in \mathcal{Q}_m$, we have on one hand $S_p 1 = p$, and on the other hand

$$S_p f = 0,$$

for all $f \in \mathcal{Q}_m$ such that $f(0) = 0$. Therefore, $\ker S_p = \mathcal{Q}_m \ominus \mathbb{C}$ or, equivalently

$$\ker S_p = \bigoplus_{t=1}^m H_t.$$

This allows us to conclude that

$$\|S_p\| = 1. \tag{3.1}$$

We recall in passing that $\|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{T}^n))} = \|\varphi\|_\infty$ for all $\varphi \in L^\infty(\mathbb{T}^n)$ (see (2.5)).

Theorem 3.2. *S_p admits a lift if and only if p is a unimodular constant multiple of a monomial.*

Proof. Suppose S_p is liftable. There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $S_p = S_\varphi$. Then

$$S_p = S_\varphi = P_{\mathcal{Q}_m} T_\varphi|_{\mathcal{Q}_m},$$

and

$$\|\varphi\|_\infty \leq 1.$$

Note that $1 \in \mathcal{Q}_m$. Since $S_p 1 = p$, it is clear that $P_{\mathcal{Q}_m} \varphi = p$, and hence there exists $\psi \in \mathcal{Q}_m^\perp$ such that

$$\varphi = p \oplus \psi \in \mathcal{Q}_m \oplus \mathcal{Q}_m^\perp.$$

It is a well known general fact that $\|\varphi\|_2 \leq \|\varphi\|_\infty$. Indeed

$$\begin{aligned} \|\varphi\|_2 &= \|T_\varphi 1\|_2 \\ &\leq \|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{T}^n))} \|1\|_2 \\ &= \|\varphi\|_\infty. \end{aligned}$$

Now that $\|p\|_2 = 1$, we compute

$$\begin{aligned} 1 + \|\psi\|_2^2 &= \|\varphi\|_2^2 \\ &\leq \|\varphi\|_\infty^2 \\ &\leq 1, \end{aligned}$$

which implies $\psi = 0$. Therefore

$$\varphi = p \in \mathcal{Q}_m.$$

By using the same computation (or the standard norm equality) as above, we have

$$1 = \|p\|_2 \leq \|p\|_\infty = \|\varphi\|_\infty = 1,$$

which implies that $\|p\|_\infty = 1$. This combined with

$$\|T_p 1\|_2 = \|p\|_2 = 1,$$

imply that the Toeplitz operator T_p is norm attaining. Consequently [28, Corollary 2.3], p is inner (as $p \in H^\infty(\mathbb{D}^n)$). Then by Lemma 3.1 we conclude that p is a unimodular constant multiple of a monomial. The converse is obvious. \square

The following corollary is now straight. Here we need to assume that $n > 1$.

Corollary 3.3. *Suppose $n > 1$. Let p be a homogeneous polynomial of degree m and assume that $\|p\|_2 = 1$. Suppose*

$$p = \sum_{|k|=m} a_k z^k \in H_m.$$

If $a_k, a_l \neq 0$ for some $k, l \in \mathbb{Z}_+^n$, then S_p on \mathcal{Q}_m is not liftable.

The following fact was used to prove the above theorem [28, Corollary 2.3]: For $\varphi \in H^\infty(\mathbb{T}^n)$ with $\|\varphi\|_\infty = 1$, if the Toeplitz operator T_φ is norm attaining, then the symbol φ is inner. In this context, it is worth noting that the lift of a commutant is highly nonunique, and the issue of uniqueness of Sarason's commutant lifting theorem is inextricably linked to the norm attainment property [60, Section 5].

Now we consider a simple class of quotient modules where all module maps admit lifting. Our idea is fairly elementary: embed one Hardy space into another Hardy space. Fix a natural number m such that $1 < m < n$. Define

$$\mathcal{S} = \sum_{j=1}^m z_j H^2(\mathbb{T}^n).$$

Then \mathcal{S} is a closed subspace and hence a submodule of $H^2(\mathbb{T}^n)$ [61]. Our interest is in the corresponding quotient module \mathcal{Q} , that is

$$\mathcal{Q} := \left(\sum_{j=1}^m z_j H^2(\mathbb{T}^n) \right)^\perp.$$

A simple calculation reveals that

$$\mathcal{Q} = \mathbb{C} \otimes H^2(\mathbb{T}^{n-m}),$$

that is, \mathcal{Q} is simply the space of functions on $H^2(\mathbb{T}^n)$ that does not depend on the variables $\{z_1, \dots, z_m\}$ (again, see [61]). Because $S_{z_i} = P_{\mathcal{Q}} T_{z_i}|_{\mathcal{Q}}$, we have

$$S_{z_i} = \begin{cases} 0 & \text{if } i = 1, \dots, m \\ T_{z_i} & \text{if } i = m+1, \dots, n. \end{cases}$$

Suppose $X \in \mathcal{B}(\mathcal{Q})$. Then, by a routine argument, X is a module map, that is

$$X P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} X,$$

for all $i = 1, \dots, n$, if and only if there exists $\varphi \in H^\infty(\mathbb{D}^n)$ such that φ does not depend on the variables $\{z_1, \dots, z_m\}$ and

$$X = T_\varphi.$$

This immediately implies the following result: Let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then $X = T_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{D}^n)$. In particular, X lifts to T_φ itself.

In Section 8, we will show examples of module maps on nonhomogeneous quotient modules that cannot be lifted.

4. CLASSIFICATIONS OF COMMUTANT LIFTING

Given the examples in the preceding section, it is clear that a module map on a quotient module of $H^2(\mathbb{T}^n)$, $n \geq 2$, may not admit a lift in general. In this section, we classify liftable module maps defined on quotient modules of $H^2(\mathbb{T}^n)$, $n \geq 1$. We begin with the well known duality of classical Banach spaces. Recall that $L^1(\mathbb{T}^n)$ is a Banach space predual of $L^\infty(\mathbb{T}^n)$. More specifically, we have

$$(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n),$$

via the isometrically isomorphic map $\chi : L^\infty(\mathbb{T}^n) \rightarrow (L^1(\mathbb{T}^n))^*$ defined by $\varphi \in L^\infty(\mathbb{T}^n) \mapsto \chi_\varphi \in (L^1(\mathbb{T}^n))^*$, where for each $\varphi \in L^\infty(\mathbb{T}^n)$, $\chi_\varphi \in (L^1(\mathbb{T}^n))^*$ is defined by

$$\chi_\varphi f = \int_{\mathbb{T}^n} \varphi f \, d\mu, \quad (4.1)$$

for all $f \in L^1(\mathbb{T}^n)$. Moreover, we have the isometric property

$$\|\chi_\varphi\| = \|\varphi\|_\infty,$$

for all $\varphi \in L^\infty(\mathbb{T}^n)$. For a nonempty $X \subseteq L^2(\mathbb{T}^n)$, we define

$$X^{conj} = \{\bar{f} : f \in X\}.$$

We also define the subspace of “mixed functions” of $L^2(\mathbb{T}^n)$ as

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{conj} + H^2(\mathbb{T}^n)).$$

This is the closed subspace of $L^2(\mathbb{T}^n)$ generated by monomials that are neither analytic nor coanalytic. Let $I_n = \{1, \dots, n\}$. Given $A \subseteq I_n$, we set

$$|A| = \#A,$$

the cardinality of A . The following easy-to-see equality explains the terminology of “mixed functions”:

$$\mathcal{M}_n = \overline{\text{span}}\{z_A^{k_A} \bar{z}_B^{k_B} : A, B \subseteq I_n, A \cap B = \emptyset, A, B \neq \emptyset, k_A \in \mathbb{Z}_+^{|A|}, k_B \in \mathbb{Z}_+^{|B|}\}, \quad (4.2)$$

where for a nonempty subset $A = \{i_1, \dots, i_m\} \subseteq I_n$ and $k_A = (k_1, \dots, k_m) \in \mathbb{Z}_+^{|A|}$, we define the monomial

$$z_A^{k_A} := z_{i_1}^{k_1} \cdots z_{i_m}^{k_m}.$$

Note that \mathcal{M}_n is self-adjoint, that is

$$\mathcal{M}_n^{conj} = \mathcal{M}_n. \quad (4.3)$$

It is also crucial to observe that if $n = 1$, then \mathcal{M}_n is trivial:

$$\mathcal{M}_1 = \{0\}. \quad (4.4)$$

Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$, as per our convention, we have $\mathcal{Q}^{conj} := \{\bar{f} : f \in \mathcal{Q}\}$, and hence \mathcal{Q}^{conj} is a closed subspace of $L^2(\mathbb{T}^n)$ and

$$\mathcal{Q}^{conj} \perp H_0^2(\mathbb{T}^n),$$

where $H_0^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) \ominus \{1\}$. It is easy to check (for instance, by using $\mathbb{S}(\cdot, 0) \equiv 1$) that

$$H_0^2(\mathbb{T}^n) = \{f \in H^2(\mathbb{T}^n) : f(0) = 0\},$$

the closed subspace of $H^2(\mathbb{T}^n)$ of functions vanishing at the origin. Finally, given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$, we set

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

The skew sums in the above definition are in fact Hilbert space orthogonal direct sums in $L^2(\mathbb{T}^n)$. However, in what follows, we will represent $\mathcal{M}_{\mathcal{Q}}$ as a linear subspace of the Banach space $L^1(\mathbb{T}^n)$, and denote it by

$$(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1).$$

We are now ready for our first lifting theorem.

Theorem 4.1. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module, let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map, and suppose $\psi = X(P_{\mathcal{Q}}1)$. Define $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \rightarrow \mathbb{C}$ by*

$$X_{\mathcal{Q}}f = \int_{\mathbb{T}^n} \psi f d\mu,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$. Then X is liftable if and only if $X_{\mathcal{Q}}$ is a contractive functional on $(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1)$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{D}^n)$ be a lift of X . Then $X = S_{\varphi}$, where, by definition, $S_{\varphi} = P_{\mathcal{Q}}T_{\varphi}|_{\mathcal{Q}}$. Since $\varphi \in \mathcal{S}(\mathbb{D}^n)$ (that is, $\|\varphi\|_{\infty} \leq 1$), it follows that the functional $\chi_{\varphi} : L^1(\mathbb{T}^n) \rightarrow \mathbb{C}$ defined by

$$\chi_{\varphi}(f) = \int_{\mathbb{T}^n} f\varphi d\mu,$$

for all $f \in L^1(\mathbb{T}^n)$, is a contraction (see (4.1)). In view of the fact that $\varphi\mathcal{Q}^{\perp} \subseteq \mathcal{Q}^{\perp}$ (as submodules are invariant under $H^{\infty}(\mathbb{D}^n)$), we have $P_{\mathcal{Q}}T_{\varphi}P_{\mathcal{Q}} = P_{\mathcal{Q}}T_{\varphi}$, and hence

$$\begin{aligned} S_{\varphi}P_{\mathcal{Q}}1 &= P_{\mathcal{Q}}T_{\varphi}|_{\mathcal{Q}}P_{\mathcal{Q}}1 \\ &= P_{\mathcal{Q}}T_{\varphi}1 \\ &= P_{\mathcal{Q}}\varphi. \end{aligned}$$

Also, $X = S_{\varphi}$ implies $\psi = S_{\varphi}P_{\mathcal{Q}}1$. This combined with $S_{\varphi}P_{\mathcal{Q}}1 = P_{\mathcal{Q}}\varphi$ yields

$$\psi = P_{\mathcal{Q}}\varphi.$$

We now prove that $X_{\mathcal{Q}}$ on $(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1)$ is a contractive functional. First we consider $X_{\mathcal{Q}}$ on $\mathcal{Q}^{conj} \subseteq \mathcal{M}_{\mathcal{Q}}$. Let $h \in \mathcal{Q}^{conj}$. Then $h \in \mathcal{Q}$ or, equivalently, $P_{\mathcal{Q}}h = h$, and we have

$$\begin{aligned} \int_{\mathbb{T}^n} \varphi \bar{h} d\mu &= \langle \varphi, h \rangle_{H^2(\mathbb{T}^n)} \\ &= \langle \varphi, P_{\mathcal{Q}}h \rangle_{H^2(\mathbb{T}^n)} \\ &= \langle P_{\mathcal{Q}}\varphi, h \rangle_{H^2(\mathbb{T}^n)} \\ &= \langle \psi, h \rangle_{H^2(\mathbb{T}^n)}. \end{aligned}$$

Thus we conclude that

$$\int_{\mathbb{T}^n} \psi \bar{h} d\mu = \int_{\mathbb{T}^n} \varphi \bar{h} d\mu,$$

for all $\bar{h} \in \mathcal{Q}^{conj}$, equivalently

$$X_{\mathcal{Q}} = \chi_{\varphi} \text{ on } \mathcal{Q}^{conj}.$$

Next, we consider $X_{\mathcal{Q}}$ on \mathcal{M}_n . Since

$$\mathcal{M}_n \subseteq L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n) + H^2(\mathbb{T}^n)^{conj}),$$

functions in \mathcal{M}_n do not have an analytic part. Moreover, since \mathcal{M}_n is self-adjoint (see (4.3)), we have

$$P_{\mathcal{Q}}\mathcal{M}_n^{conj} = P_{\mathcal{Q}}\mathcal{M}_n = \{0\}.$$

By using the identity $\psi = P_{\mathcal{Q}}\varphi$ and following the computation as in the previous case, for each $h \in \mathcal{M}_n$, we have

$$\begin{aligned} \int_{\mathbb{T}^n} \psi h \, d\mu &= \langle \psi, \bar{h} \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle P_{\mathcal{Q}}\varphi, \bar{h} \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle \varphi, P_{\mathcal{Q}}\bar{h} \rangle_{H^2(\mathbb{T}^n)} \\ &= 0, \end{aligned}$$

as $P_{\mathcal{Q}}h = P_{\mathcal{Q}}\bar{h} = 0$. This proves that

$$X_{\mathcal{Q}} = \chi_{\varphi} = 0 \text{ on } \mathcal{M}_n.$$

Finally, if $h \in H_0^2(\mathbb{T}^n)$, then $h(0) = 0$, and hence (as $\psi \in \mathcal{Q} \subseteq H^2(\mathbb{T}^n)$)

$$\langle \bar{h}, \psi \rangle_{L^2(\mathbb{T}^n)} = 0.$$

Therefore, again

$$\begin{aligned} \int_{T^n} \psi h \, d\mu &= \langle \psi, \bar{h} \rangle_{L^2(\mathbb{T}^n)} \\ &= 0, \end{aligned}$$

as $\psi \in \mathcal{Q} \subset H^2(\mathbb{T}^n)$. This implies, again, that

$$X_{\mathcal{Q}} = \chi_{\varphi} = 0 \text{ on } H_0^2(\mathbb{T}^n).$$

Thus we conclude that $X_{\mathcal{Q}} = \chi_{\varphi}$ on $\mathcal{M}_{\mathcal{Q}}$. On the other hand, $\chi_{\varphi} : L^1(\mathbb{T}^n) \rightarrow \mathbb{C}$ is a contraction. In particular, $\chi_{\varphi}|_{\mathcal{M}_{\mathcal{Q}}}$ is a contraction, which proves our claim that $X_{\mathcal{Q}} : \mathcal{M}_{\mathcal{Q}} \rightarrow \mathbb{C}$ is contractive.

For the converse direction, assume that $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \rightarrow \mathbb{C}$ is a contraction. By the Hahn-Banach theorem, there is a linear functional $\tilde{X}_{\mathcal{Q}} : L^1(\mathbb{T}^n) \rightarrow \mathbb{C}$ such that

$$\tilde{X}_{\mathcal{Q}}|_{\mathcal{M}_{\mathcal{Q}}} = X_{\mathcal{Q}},$$

and

$$\|\tilde{X}_{\mathcal{Q}}\| = \|X_{\mathcal{Q}}\| \leq 1.$$

By the duality $(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n)$, as outlined in (4.1), there exists $\varphi \in L^\infty(\mathbb{T}^n)$ such that

$$\chi_{\varphi} = \tilde{X}_{\mathcal{Q}},$$

and

$$\|\varphi\|_{\infty} \leq 1.$$

In particular, $\chi_\varphi|_{\mathcal{M}_Q} = \tilde{X}_Q|_{\mathcal{M}_Q} = X_Q$. Since

$$\chi_\varphi h = \int_{\mathbb{T}^n} \varphi h d\mu,$$

for all $h \in \mathcal{M}_Q$, it follows that

$$\int_{\mathbb{T}^n} \varphi h d\mu = \int_{\mathbb{T}^n} \psi h d\mu, \quad (4.5)$$

for all $h \in \mathcal{M}_Q$. We consider a typical monomial f from $\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)$. In other words, we let

$$f = z^k,$$

for some $k \in \mathbb{N}^n$, or let

$$f = z_A^{k_A} \bar{z}_B^{k_B},$$

for some $k_A \in \mathbb{Z}_+^{|A|}$ and $k_B \in \mathbb{Z}_+^{|B|}$, where $A, B \subseteq \{1, \dots, n\}$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$ (see the definition of \mathcal{M}_n in (4.2)). As $\psi = X(P_Q 1) \in \mathcal{Q} \subseteq \text{Hol}(\mathbb{D}^n)$, it follows that $\langle \psi, \bar{f} \rangle_{L^2(\mathbb{T}^n)} = 0$ and hence

$$\int_{\mathbb{T}^n} \psi f d\mu = 0.$$

Consequently, the identity in (4.5) yields

$$\int_{\mathbb{T}^n} \varphi z^k d\mu = 0,$$

for all $k \in \mathbb{N}^n$, as well as

$$\int_{\mathbb{T}^n} \varphi z_A^{k_A} \bar{z}_B^{k_B} d\mu = 0,$$

for all $k_A \in \mathbb{Z}_+^{|A|}$ and $k_B \in \mathbb{Z}_+^{|B|}$, where $A, B \subseteq \{1, \dots, n\}$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$. This implies φ is analytic, and hence $\varphi \in \mathcal{S}(\mathbb{D}^n)$. To complete the proof, it remains to show that $X = S_\varphi$. Note, by (4.5) again, we have that

$$\int_{\mathbb{T}^n} \psi \bar{h} d\mu = \int_{\mathbb{T}^n} \varphi \bar{h} d\mu,$$

for all $\bar{h} \in \mathcal{Q}^{conj}$. Equivalently, for each $\bar{h} \in \mathcal{Q}^{conj}$, we have

$$\langle \varphi, h \rangle_{L^2(\mathbb{T})} = \langle \psi, h \rangle_{L^2(\mathbb{T}^n)},$$

and hence

$$\langle P_Q \varphi, h \rangle_{H^2(\mathbb{T}^n)} = \langle \psi, h \rangle_{H^2(\mathbb{T}^n)},$$

from which we conclude that

$$P_Q \varphi = \psi.$$

As before, we write $\varphi \in \mathcal{S}(\mathbb{D}^n) \subseteq H^2(\mathbb{T}^n)$ with respect to $\mathcal{Q} \oplus \mathcal{Q}^\perp = H^2(\mathbb{T}^n)$ as

$$\varphi = \psi \oplus \rho \in \mathcal{Q} \oplus \mathcal{Q}^\perp.$$

Since $P_Q \varphi = P_Q T_\varphi P_Q 1 = S_\varphi(P_Q 1)$ and $P_Q \varphi = \psi$, we have

$$\psi = S_\varphi(P_Q 1).$$

This combined with $\psi = X(P_Q 1)$ yields

$$S_\varphi(P_Q 1) = X(P_Q 1).$$

Finally, let us fix $k \in \mathbb{Z}_+^n$ and observe

$$\begin{aligned} P_Q z^k &= P_Q z^k (P_Q 1) \\ &= S_z^k (P_Q 1). \end{aligned}$$

Therefore, $S_\varphi S_z^k = S_z^k S_\varphi$ implies

$$\begin{aligned} S_\varphi (P_Q z^k) &= S_\varphi S_z^k (P_Q 1) \\ &= S_z^k S_\varphi (P_Q 1) \\ &= S_z^k X (P_Q 1) \\ &= X S_z^k (P_Q 1) \\ &= X (P_Q z^k). \end{aligned}$$

Then, in view of the fact that $\mathcal{Q} = \overline{\text{span}}\{P_Q z^k : k \in \mathbb{Z}_+^n\}$, the equality $X = S_\varphi$ is immediate. This completes the proof of the theorem. \square

The proof of the above theorem says more than what it states. In fact, we have the identity

$$X_Q|_{\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)} \equiv 0,$$

and hence

$$\ker X_Q \supseteq \mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n).$$

In other words, \mathcal{Q}^{conj} is the supporting space of the functional X_Q . Another way to put it is that there is a contractive extension of $X_Q|_{\mathcal{Q}^{conj}}$ to the entire \mathcal{M}_Q that vanishes on the completely analytic and completely co-analytic parts.

Remark 4.2. It is clear from the construction that the subspace $(\mathcal{M}_Q, \|\cdot\|_1)$ is independent of X .

Our second lifting theorem is a consequence of the first, and it appears to be in a more compact form. Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ and a module map $X \in \mathcal{B}(\mathcal{Q})$, we define a subspace of $L^1(\mathbb{T}^n)$ as

$$\tilde{\mathcal{M}}_{\mathcal{Q},X} = (\mathcal{Q}^{conj} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Keep in mind, in contrast to Remark 4.2, that $(\tilde{\mathcal{M}}_{\mathcal{Q},X}, \|\cdot\|_1)$ is dependent on X .

Theorem 4.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module, let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map, and suppose $\psi = X(P_Q 1)$. Then X is liftable if and only if*

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q},X}\right) \geq 1.$$

Proof. In view of $\bar{\psi} \in \mathcal{Q}^{conj}$, first we observe that

$$\mathcal{M}_Q = \mathbb{C}\bar{\psi} \dot{+} \tilde{\mathcal{M}}_{\mathcal{Q},X}.$$

Suppose X is liftable. By Theorem 4.1, we have

$$\left| \int_{\mathbb{T}^n} \psi f d\mu \right| \leq \|f\|_1 \quad (f \in \mathcal{M}_Q). \quad (4.6)$$

Pick $g \in \mathcal{M}_Q$. There exist a scalar c and a function $\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$ such that

$$g = c\bar{\psi} + \tilde{g}.$$

We compute

$$\begin{aligned} \int_{\mathbb{T}^n} \psi(c\bar{\psi} + \tilde{g}) d\mu &= c \int_{\mathbb{T}^n} \psi\bar{\psi} d\mu + \int_{\mathbb{T}^n} \psi\tilde{g} d\mu \\ &= c\|\psi\|_2^2 + \langle \psi, \bar{\tilde{g}} \rangle \\ &= c\|\psi\|_2^2, \end{aligned}$$

as

$$\langle \psi, \bar{\tilde{g}} \rangle = 0,$$

which follows from the definition of \tilde{g} and the fact that ψ is analytic. Now (4.6) implies

$$\left| \int_{\mathbb{T}^n} \psi(c\bar{\psi} + \tilde{g}) d\mu \right| \leq \|c\bar{\psi} + \tilde{g}\|_1,$$

and hence

$$|c|\|\psi\|_2^2 \leq \|c\bar{\psi} + \tilde{g}\|_1,$$

or equivalently

$$\left\| \frac{\bar{\psi}}{\|\psi\|_2^2} + \tilde{g} \right\|_1 \geq 1,$$

for all $\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$, and completes the proof of the forward direction. To prove the reverse direction, let the above inequality holds for all $\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$. Equivalently

$$\|\psi\|_2^2 \leq \|\bar{\psi} + \tilde{g}\|_1 \quad (\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}).$$

Fix $f \in \mathcal{M}_{\mathcal{Q}}$, and write $f = c\bar{\psi} + \tilde{f}$ for some scalar c and some function $\tilde{f} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$. Following the proof of the forward direction, we have

$$\begin{aligned} c\|\psi\|_2^2 &= \int_{\mathbb{T}^n} \psi(c\bar{\psi} + \tilde{f}) d\mu \\ &= c \int_{\mathbb{T}^n} \psi f d\mu, \end{aligned}$$

which leads to (4.6). Theorem 4.1 now completes the proof of the theorem. \square

Combining Theorem 4.1 and Theorem 4.3, we have the following:

Theorem 4.4. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Set*

$$\psi = X(P_{\mathcal{Q}}1),$$

and suppose

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)),$$

and

$$\tilde{\mathcal{M}}_{\mathcal{Q},X} = (\mathcal{Q}^{conj} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Then the following conditions are equivalent:

- (1) X is liftable.
- (2) $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \longrightarrow \mathbb{C}$ is a contractive functional, where

$$X_{\mathcal{Q}}f = \int_{\mathbb{T}^n} \psi f d\mu \quad (f \in \mathcal{M}_{\mathcal{Q}}).$$

- (3) $\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q},X}\right) \geq 1.$

The techniques involved in the association of the existence of commutant lifting with the distance formula are far-reaching. In the following section, we will apply some of the concepts introduced here to solve a perturbation problem.

5. PERTURBATIONS OF ANALYTIC FUNCTIONS

Our aim is to present a classification of $H^2(\mathbb{T}^n)$ -functions that can be perturbed by $H^2(\mathbb{T}^n)$ -functions so that the resultant functions are in $\mathcal{S}(\mathbb{D}^n)$. Our perturbation result is of independent interest and not directly related to the commutant lifting theorem. However, the technique involved here is motivated by the one used in the proof of our main results.

Throughout the sequel, we denote

$$\mathcal{L}_n = \mathcal{M}_n \oplus H_0^2(\mathbb{T}^n).$$

Recall that $H_0^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) \ominus \{1\}$ is the closed subspace of $H^2(\mathbb{T}^n)$ of functions vanishing at the origin. Recall also that

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{conj} + H^2(\mathbb{T}^n)),$$

the closed subspace of $L^2(\mathbb{T}^n)$ generated by all the trigonometric monomials that are neither analytic nor co-analytic. In particular, we have the crucial property that

$$\langle f, 1 \rangle_{L^2(\mathbb{T}^n)} = 0 \quad (f \in \mathcal{L}_n).$$

Finally, we recall a basic fact from Banach space theory: Let x be a vector in a Banach space B . Then

$$\|x\|_B = \sup\{|x^*(x)| : x^* \in B^*, \|x^*\| \leq 1\}.$$

Now we are ready for the perturbation theorem.

Theorem 5.1. *Let $f \in H^2(\mathbb{T}^n)$ be a nonzero function. There exists $g \in \{f\}^\perp$ such that $f + g \in \mathcal{S}(\mathbb{D}^n)$ if and only if*

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{f}}{\|f\|_2^2}, \mathcal{L}_n\right) \geq 1.$$

Proof. We start by recalling the definition of distance function (in the present case):

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{f}}{\|f\|_2^2}, \mathcal{L}_n\right) = \inf \left\{ \left\| \frac{\bar{f}}{\|f\|_2^2} + h \right\|_1 : h \in \mathcal{L}_n \right\}.$$

Suppose $g \in \{f\}^\perp$ be such that $\psi := f + g \in \mathcal{S}(\mathbb{D}^n)$. It is enough to prove that

$$\left\| \frac{\bar{f}}{\|f\|_2^2} + h \right\|_1 \geq 1 \quad (h \in \mathcal{L}_n).$$

Fix $h \in \mathcal{L}_n$. Since $\psi \in \mathcal{S}(\mathbb{D}^n)$ and $\mathcal{S}(\mathbb{D}^n)$ is a subset of the closed unit ball of $L^\infty(\mathbb{T}^n)$, we have $\psi \in L^\infty(\mathbb{T}^n)$ and $\|\psi\|_\infty \leq 1$. By the duality (see (4.1))

$$(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n),$$

it follows that $\chi_\psi \in (L^1(\mathbb{T}^n))^*$ and

$$\|\psi\|_\infty = \|\chi_\psi\| \leq 1,$$

where

$$\chi_\psi g = \int_{\mathbb{T}^n} \psi g \, d\mu,$$

for all $g \in L^1(\mathbb{T}^n)$. In particular, for

$$g = \frac{\bar{f}}{\|f\|_2^2} + h \in L^1(\mathbb{T}^n),$$

we compute

$$\begin{aligned} \int_{\mathbb{T}^n} \psi\left(\frac{\bar{f}}{\|f\|_2^2} + h\right) d\mu &= \left\langle f + g, \frac{f}{\|f\|_2^2} + \bar{h} \right\rangle_{L^2(\mathbb{T}^n)} \\ &= 1 + \left\langle g, \frac{f}{\|f\|_2^2} + \bar{h} \right\rangle_{L^2(\mathbb{T}^n)} \\ &= 1. \end{aligned}$$

The last but one equality follows from the fact that (note that \bar{h} has no analytic part)

$$\langle f, \bar{h} \rangle_{L^2(\mathbb{T}^n)} = 0,$$

and the last equality is due to the fact that $g \in \{f\}^\perp$ and

$$\langle g, \bar{h} \rangle_{L^2(\mathbb{T}^n)} = 0,$$

similar reason as in the preceding equality. We also have used the fact that f is analytic and $\langle h, 1 \rangle_{L^2(\mathbb{T}^n)} = 0$. Therefore, $\chi_\psi \in (L^1(\mathbb{T}^n))^*$ with $\|\chi_\psi\| \leq 1$ and

$$\left| \chi_\psi\left(\frac{\bar{f}}{\|f\|_2^2} + h\right) \right| = 1.$$

The norm identity for Banach spaces stated preceding the statement of this theorem immediately implies that

$$\left\| \frac{\bar{f}}{\|f\|_2^2} + h \right\|_1 \geq 1.$$

For the reverse direction, suppose the above inequality holds for all $h \in \mathcal{L}_n$. Equivalently

$$\|\lambda \bar{f} + h\|_1 \geq |\lambda| \|f\|_2^2,$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. Define \mathcal{S} a subspace of $L^1(\mathbb{T}^n)$ as

$$\mathcal{S} := \text{span}\{\bar{f}, \mathcal{L}_n\},$$

and then define a linear functional $\zeta_f : \mathcal{S} \rightarrow \mathbb{C}$ by

$$\zeta_f(\lambda \bar{f} + h) = \int_{\mathbb{T}^n} (\lambda \bar{f} + h) f d\mu,$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. As in the proof of the forward direction, we have

$$\begin{aligned} \int_{\mathbb{T}^n} f h d\mu &= \langle h, \bar{f} \rangle_{L^2(\mathbb{T}^n)} \\ &= 0, \end{aligned}$$

for all $h \in \mathcal{L}_n$. Moreover, since

$$\int_{\mathbb{T}^n} f \bar{f} d\mu = \|f\|_2^2,$$

it follows that

$$\begin{aligned} |\zeta_f(\lambda \bar{f} + h)| &= |\lambda| \|f\|_2^2 \\ &\leq \|\lambda \bar{f} + h\|_1, \end{aligned}$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. This ensures that ζ_f is a contractive functional on \mathcal{S} ; hence, by the Hahn-Banach theorem, there exists $\zeta \in (L^\infty(\mathbb{T}^n))^*$ such that $\|\zeta\| \leq 1$ and

$$\zeta|_{\mathcal{S}} = \zeta_f.$$

Again, by the duality (4.1), there exists $\varphi \in L^\infty(\mathbb{T}^n)$ such that $\|\varphi\|_\infty \leq 1$ and

$$\chi_\varphi|_{\mathcal{S}} = \zeta|_{\mathcal{S}} = \zeta_f.$$

Therefore

$$\int_{\mathbb{T}^n} (\lambda \bar{f} + h) f d\mu = \int_{\mathbb{T}^n} (\lambda \bar{f} + h) \varphi d\mu, \quad (5.1)$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. We now claim that φ is analytic (which would clearly imply that $\varphi \in H^\infty(\mathbb{D}^n)$). As in the proof of Theorem 4.1, we consider a typical monomial F from $\mathcal{L}_n = \mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)$. Therefore

$$F = z^k,$$

for some $k \in \mathbb{N}^n$, or

$$F = z_A^{k_A} \bar{z}_B^{k_B},$$

for some $k_A \in \mathbb{Z}_+^{|A|}$ and $k_B \in \mathbb{Z}_+^{|B|}$, where $A, B \subseteq \{1, \dots, n\}$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$ (see the representation of \mathcal{M}_n in (4.2)). We compute

$$\begin{aligned} 0 &= \langle f, \bar{F} \rangle_{L^2(\mathbb{T}^n)} \\ &= \int_{\mathbb{T}^n} f \bar{F} d\mu \\ &= \int_{\mathbb{T}^n} \varphi F d\mu \\ &= \langle \varphi, \bar{F} \rangle_{L^2(\mathbb{T}^n)}, \end{aligned}$$

which proves the claim. Since $\|\varphi\|_\infty \leq 1$, we conclude that $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Using the containment $H^\infty(\mathbb{D}^n) \subseteq H^2(\mathbb{D}^n)$, first we conclude $\varphi \in H^2(\mathbb{D}^n)$, and then write

$$\varphi = cf \oplus g,$$

for some scalar c and function $g \in H^2(\mathbb{D}^n)$ such that $g \in \{f\}^\perp$. It remains to show that $c = 1$. Observe, if $h = 0$, and

$$\lambda = \frac{1}{\|f\|_2^2},$$

then (5.1) along with the fact that $\langle g, f \rangle = 0$ yields

$$\begin{aligned} 1 &= \int_{\mathbb{T}^n} f \frac{\bar{f}}{\|f\|_2^2} d\mu \\ &= \int_{\mathbb{T}^n} \varphi \frac{\bar{f}}{\|f\|_2^2} d\mu \\ &= \left\langle \varphi, \frac{f}{\|f\|_2^2} \right\rangle_{H^2(\mathbb{T}^n)} \\ &= \left\langle cf \oplus g, \frac{f}{\|f\|_2^2} \right\rangle_{H^2(\mathbb{T}^n)} \\ &= c. \end{aligned}$$

This completes the proof of the theorem. □

We know, in particular, that $\mathcal{M}_1 = \{0\}$ (see (4.4)). Moreover, as observed earlier, that $H_0^2(\mathbb{T}) = zH^2(\mathbb{T})$. Therefore

$$\mathcal{L}_1 = zH^2(\mathbb{T}),$$

and as a result, the preceding theorem is simplified as follows:

Corollary 5.2. *Given $f \neq 0$ in $H^2(\mathbb{T})$, there exists $g \in \{f\}^\perp$ such that*

$$f + g \in \mathcal{S}(\mathbb{D}),$$

if and only if

$$\text{dist}_{L^1(\mathbb{T})}\left(\frac{\bar{f}}{\|f\|_2^2}, zH^2(\mathbb{T})\right) \geq 1.$$

The following example illustrates the above theorem.

Example 5.3. Fix a real number $0 < c < 1$, and pick $b \in (c^2, c)$. Also fix a multiindex $k_0 \in \mathbb{Z}_+^n$, $k_0 \neq (0, \dots, 0)$, and set

$$\Lambda := \mathbb{Z}_+^n \setminus \{k_0\}.$$

Finally, choose a sequence $\{a_k\}_{k \in \Lambda} \subseteq \mathbb{R}_+$ such that

- (1) $\sum_{k \in \Lambda} a_k$ diverges, and
- (2) $\sqrt{\sum_{k \in \Lambda} a_k^2 + b^2} = c.$

Set

$$f = \sum_{k \in \Lambda} a_k z^k + b z^{k_0}.$$

We want to show that f can be perturbed to become a Schur function. To this end, we first observe that $f(0, \dots, 0) = a_0$ and

$$f(1, \dots, 1) = b + \sum_{k \in \Lambda} a_k,$$

and hence (by continuity)

$$f(L) = (a_0, \infty),$$

where L is the line joining $(0, \dots, 0)$ and $(1, \dots, 1)$. We conclude, in particular, that

$$f \notin H^\infty(\mathbb{D}^n).$$

Moreover

$$\|f\|_2 = c,$$

by construction of f . We now consider the functional $\chi_{z^{k_0}} \in (L^1(\mathbb{T}^n))^*$ (see the duality (4.1)). Clearly

$$\|\chi_{z^{k_0}}\| = 1.$$

Given arbitrary functions $g \in \mathcal{M}_n$ and $h \in H_0^2(\mathbb{T}^n)$, we compute

$$\begin{aligned} \chi_{z^{k_0}}\left(\frac{\bar{f}}{c^2} + g + h\right) &= \int_{\mathbb{T}^n} z^{k_0} \left(\frac{\bar{f}}{c^2} + g + h\right) d\mu \\ &= \left\langle \frac{\bar{f}}{c^2} + g + h, \bar{z}^{k_0} \right\rangle_{L^2(\mathbb{T}^n)} \\ &= \left\langle \frac{\bar{f}}{c^2}, \bar{z}^{k_0} \right\rangle_{L^2(\mathbb{T}^n)} \\ &= \frac{b}{c^2}. \end{aligned}$$

Since $b > c^2$, it follows that

$$\chi_{z^{k_0}}\left(\frac{\bar{f}}{c^2} + g + h\right) \geq 1,$$

and consequently, the norm identity that was mentioned preceding the statement of Theorem 6.6 infers that

$$\left\| \frac{\bar{f}}{c^2} + g + h \right\|_1 \geq 1.$$

Given that $\mathcal{L}_n = \mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)$, the above is equivalent to saying that

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{f}}{\|\bar{f}\|_2^2}, \mathcal{L}_n\right) \geq 1,$$

and hence, by Theorem 5.1, we conclude that $f \oplus g \in \mathcal{S}(\mathbb{D}^n)$ for some $g \in \{f\}^\perp$.

It may appear to be a coincidence that the distance recipe in Theorem 5.1 as well as in Theorem 4.3 (and the quantitative interpolation theorem in Section 7) is similar to the well known Nehari theorem [49] for Hankel operators. Recall that the Hankel operator with symbol $\varphi \in L^\infty(\mathbb{T})$ is defined by

$$H_\varphi = P_{H_-^2(\mathbb{T})} L_\varphi|_{H^2(\mathbb{T})},$$

where $H_-^2(\mathbb{T}) = L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. The Nehari theorem states:

$$\|H_\varphi\| = \text{dist}(\varphi, H^\infty(\mathbb{D})) = \|\varphi\|_\infty.$$

Furthermore, it is well known that the Nehari problem is related to the Nevanlinna-Pick interpolation problem for rational functions. See also the well known Adamyan, Arov, and Krein theorem, also known as the AAK step-by-step extension [51, Chapter 2]. Another important formula is due to Adamyan, Arov and Krein [4]:

$$\|H_\varphi\|_{ess} = \text{dist}(\varphi, C(\mathbb{T}) + H^\infty(\mathbb{D})),$$

for all $\varphi \in L^\infty(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of all continuous functions on \mathbb{T} . Hankel operators in several variables [55] are also complex objects. We refer the reader to Coifman, Rochberg, and Weiss [18] for some progress to the theory of Hankel operators (also see [33]).

6. INTERPOLATION

The goal of this section is to provide a solution to the interpolation problem. As previously mentioned, Sarason's commuting lifting theorem recovers the Nevanlinna-Pick interpolation with an elegant proof. However, Sarason only needed to use his lifting theorem for some special finite-dimensional quotient modules. These quotient modules are generated by finitely many kernel functions.

First, we prove that Sarason-type quotient modules (we call them zero-based quotient modules) in several variables always admit lifting to $H^\infty(\mathbb{D}^n)$ -functions (we call it weak lifting).

Definition 6.1. Let $\mathcal{Q} \subseteq H^2(\mathbb{D}^n)$ be a quotient module, and let $X \in \mathcal{B}(\mathcal{Q})$. Suppose $XS_{z_i} = S_{z_i}X$ for all $i = 1, \dots, n$. We say that X admits a weak lift or X is weakly liftable if there exists $\varphi \in H^\infty(\mathbb{D}^n)$ such that

$$X = S_\varphi.$$

To put it another way, a weak lifting is a lifting that lacks control over the norm. Given a set $\mathcal{Z} \subseteq \mathbb{D}^n$, define

$$\mathcal{Q}_{\mathcal{Z}} = \overline{\text{span}}\{\mathbb{S}(\cdot, w) : w \in \mathcal{Z}\}.$$

Definition 6.2. A quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ is said to be *zero-based* if there exists $\mathcal{Z} \subseteq \mathbb{D}^n$ such that $\mathcal{Q} = \mathcal{Q}_{\mathcal{Z}}$.

For a zero-based quotient module $\mathcal{Q}_{\mathcal{Z}}$, by using the reproducing property (2.4), we have the following representation of the corresponding submodule (hence the name zero-based)

$$\mathcal{Q}_{\mathcal{Z}}^\perp = \{f \in H^2(\mathbb{T}^n) : f(w) = 0 \text{ for all } w \in \mathcal{Z}\}.$$

Since $\{\mathbb{S}(\cdot, w) : w \in \mathcal{Z}\}$ is a set of linearly independent vectors, a zero-based quotient module $\mathcal{Q}_{\mathcal{Z}}$ is finite-dimensional if and only if

$$\#\mathcal{Z} = \dim \mathcal{Q}_{\mathcal{Z}} < \infty.$$

For each $j \in \{1, \dots, n\}$, denote by $\pi_j : \mathbb{C}^n \rightarrow \mathbb{C}$ the projection map onto the j -th coordinate. In particular, $z \in \mathbb{C}^n$ can be expressed as

$$z = (\pi_1(z), \dots, \pi_n(z)).$$

The following easy-to-see lemma will be useful in what follows.

Lemma 6.3. Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be a set of distinct points, and let $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$. Then X is module map if and only if there exists $\{w_i\}_{i=1}^m \subset \mathbb{C}$ such that

$$X^*\mathbb{S}(\cdot, z_j) = w_j\mathbb{S}(\cdot, z_j),$$

for all $j = 1, \dots, m$.

Proof. Let $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ and suppose $XS_{z_i} = S_{z_i}X$ for all $i = 1, \dots, n$. Since $X^*S_{z_i}^* = S_{z_i}^*X^*$, using the fact that $\mathcal{Q}_{\mathcal{Z}}$ is a quotient module, we find

$$T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}}X^* = X^*T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}},$$

for all $i = 1, \dots, n$. In view of (2.3), we compute

$$\begin{aligned} (T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}}X^*)\mathbb{S}(\cdot, z_j) &= (X^*T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}})\mathbb{S}(\cdot, z_j) \\ &= X^*T_{z_i}^*\mathbb{S}(\cdot, z_j) \\ &= \overline{\pi_i(z_j)}X^*\mathbb{S}(\cdot, z_j). \end{aligned}$$

Since $(T_{z_i}^*|_{\mathcal{Q}_Z} X^*)\mathbb{S}(\cdot, z_j) = T_{z_i}^*(X^*\mathbb{S}(\cdot, z_j))$, it follows that

$$T_{z_i}^*(X^*\mathbb{S}(\cdot, z_j)) = \overline{\pi_i(z_j)}(X^*\mathbb{S}(\cdot, z_j)),$$

for all $i = 1, \dots, n$, and $j = 1, \dots, m$. Equivalently

$$X^*\mathbb{S}(\cdot, z_j) \in \bigcap_{i=1}^n \ker(T_{z_i} - \pi_i(z_j)I_{H^2(\mathbb{T}^n)})^*,$$

for all $j = 1, \dots, m$. Now, in view of the joint eigenspace property (2.2), the right side of the above is $\mathbb{C}\mathbb{S}(\cdot, z_j)$, and hence, there exists a scalar w_j such that

$$X^*\mathbb{S}(\cdot, z_j) = w_j\mathbb{S}(\cdot, z_j),$$

for all $j = 1, \dots, m$. The converse direction is easy and follows again from (2.3) and the definition of \mathcal{Q}_Z . \square

The proposition that follows is very crucial and will be used in what follows.

Proposition 6.4. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module. Let*

$$\theta_{\mathcal{Q}} = P_{\mathcal{Q}}1.$$

If $\theta_{\mathcal{Q}} \in H^\infty(\mathbb{D}^n)$, then $S_{\theta_{\mathcal{Q}}} = I_{\mathcal{Q}}$.

Proof. Since $\theta_{\mathcal{Q}} = P_{\mathcal{Q}}1 \in \mathcal{Q} \cap H^\infty(\mathbb{D}^n)$, in view of the decomposition $H^2(\mathbb{T}^n) = \mathcal{Q} \oplus \mathcal{Q}^\perp$, there exists $\varphi \in H^\infty(\mathbb{D}^n) \cap \mathcal{Q}^\perp$ such that

$$1 = \theta_{\mathcal{Q}} \oplus \varphi \in \mathcal{Q} \oplus \mathcal{Q}^\perp.$$

Fix $f \in \mathcal{Q}$. In particular, since $f \in H^2(\mathbb{T}^n)$, there exists a sequence

$$\{p_j\}_{j=1}^\infty \subseteq \mathbb{C}[z_1, \dots, z_n],$$

such that

$$p_j \longrightarrow f \quad \text{in } H^2(\mathbb{T}^n).$$

Since $\varphi \in H^\infty(\mathbb{D}^n) \cap \mathcal{Q}^\perp$ is a multiplier, the above implies

$$\varphi p_j \longrightarrow \varphi f \quad \text{in } H^2(\mathbb{T}^n).$$

Moreover, $\varphi \in \mathcal{Q}^\perp$ implies that

$$\{p_j \varphi\}_{j=1}^\infty \subseteq \mathcal{Q}^\perp,$$

as \mathcal{Q}^\perp is a submodule, and hence $\varphi f \in \mathcal{Q}^\perp$. Equivalently, we have

$$P_{\mathcal{Q}}(\varphi f) = 0.$$

Finally, since $\theta_{\mathcal{Q}}, \varphi \in H^\infty(\mathbb{D}^n)$, it follows that

$$\begin{aligned} f &= \theta_{\mathcal{Q}}f + \varphi f \\ &= P_{\mathcal{Q}}(\theta_{\mathcal{Q}}f + \varphi f) \quad (\text{as } f \in \mathcal{Q}) \\ &= P_{\mathcal{Q}}(\theta_{\mathcal{Q}}f) + 0 \\ &= (P_{\mathcal{Q}}(P_{\mathcal{Q}}1)|_{\mathcal{Q}})f, \end{aligned}$$

which yields $S_{\theta_{\mathcal{Q}}}f = f$, and completes the proof of the proposition. \square

We are now ready for the weak lifting. It asserts, in essence, that a module map on a finite-dimensional zero-based quotient module always admits a lift to $H^\infty(\mathbb{D}^n)$.

Corollary 6.5. *Let $\mathcal{Z} = \{z_1, \dots, z_m\} \subset \mathbb{D}^n$ be m distinct points, and let $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$. Then*

$$XS_{z_i} = S_{z_i}X,$$

for all $i = 1, \dots, m$, if and only if there exists $\varphi \in H^\infty(\mathbb{D}^n)$ such that

$$X = S_\varphi.$$

Moreover, the function φ is given by

$$\varphi = X(P_{\mathcal{Q}_{\mathcal{Z}}}1),$$

and, in particular

$$\varphi \in H^\infty(\mathbb{D}^n) \cap \mathcal{Q}_{\mathcal{Z}}.$$

Proof. The sufficient part is trivial. We prove the necessary part. For simplicity of notation, we set $\mathcal{Q} = \mathcal{Q}_{\mathcal{Z}}$. Suppose $X \in \mathcal{B}(\mathcal{Q})$ and suppose that $XS_{z_i} = S_{z_i}X$ for all $i = 1, \dots, m$. As in Proposition 6.4, set

$$\theta_{\mathcal{Q}} = P_{\mathcal{Q}}1.$$

As observed earlier, $\mathbb{S}(\cdot, w) \in H^\infty(\mathbb{D}^n)$ for all $w \in \mathbb{D}^n$ implies that $\mathcal{Q} \subseteq H^\infty(\mathbb{D}^n)$. In particular, $\theta_{\mathcal{Q}} \in H^\infty(\mathbb{D}^n)$. By Proposition 6.4, we have

$$S_{\theta_{\mathcal{Q}}} = I_{\mathcal{Q}}.$$

Since $X \in \mathcal{B}(\mathcal{Q})$, it follows that

$$\varphi := X\theta_{\mathcal{Q}} \in H^\infty(\mathbb{D}^n).$$

Therefore

$$\begin{aligned} S_\varphi\theta_{\mathcal{Q}} &= P_{\mathcal{Q}}(\varphi\theta_{\mathcal{Q}}) \\ &= S_{\theta_{\mathcal{Q}}}\varphi \\ &= \varphi \\ &= X\theta_{\mathcal{Q}}. \end{aligned}$$

The remainder of the proof is based on the standard property of the module map X . Indeed, we first observe that

$$\mathcal{Q} = \overline{\text{span}}\{P_{\mathcal{Q}}z^kP_{\mathcal{Q}}1 : k \in \mathbb{Z}_+^n\}.$$

On the other hand, for $k \in \mathbb{Z}_+^n$, since $XS_z^k = S_z^kX$, we have

$$\begin{aligned} X(P_{\mathcal{Q}}(z^k\theta_{\mathcal{Q}})) &= X(S_z^k\theta_{\mathcal{Q}}) \\ &= S_z^kX\theta_{\mathcal{Q}} \\ &= P_{\mathcal{Q}}z^k\varphi \\ &= P_{\mathcal{Q}}z^kS_\varphi\theta_{\mathcal{Q}} \\ &= S_\varphi(P_{\mathcal{Q}}(z^k\theta_{\mathcal{Q}})). \end{aligned}$$

This completes the proof of the fact that $X = S_\varphi$. The final assertion follows from the definition of $\theta_{\mathcal{Q}}$. \square

As already pointed out, the weak lifting does not touch the delicate structure of the Schur functions on \mathbb{D}^n , $n > 1$.

We will now look at the interpolation problem. Recall once again that

$$\mathbb{S}(z, w) = \prod_{i=1}^m \frac{1}{1 - z_i \bar{w}_i} \quad (z, w \in \mathbb{D}^n),$$

is the Szegő kernel of \mathbb{D}^n , and

$$\mathbb{S}(z, w) = \left\langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \right\rangle_{H^2(\mathbb{T}^n)} \quad (z, w \in \mathbb{D}^n).$$

Theorem 6.6. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. Set*

$$\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}} = \mathcal{Q}_{\mathcal{Z}}^{\text{conj}} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if

$$\mathfrak{I}_{\mathcal{Z}, \mathcal{W}} f = \int_{\mathbb{T}^n} \psi_{\mathcal{Z}, \mathcal{W}} f d\mu \quad (f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}),$$

defines a contraction $\mathfrak{I}_{\mathcal{Z}, \mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \rightarrow \mathbb{C}$, where

$$\psi_{\mathcal{Z}, \mathcal{W}} = \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i),$$

and the scalar coefficients $\{c_i\}_{i=1}^m$ are given by the identity

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \mathbb{S}(z_1, z_1) & \mathbb{S}(z_1, z_2) & \cdots & \mathbb{S}(z_1, z_m) \\ \mathbb{S}(z_2, z_1) & \mathbb{S}(z_2, z_2) & \cdots & \mathbb{S}(z_2, z_m) \\ \vdots & & \ddots & \vdots \\ \mathbb{S}(z_m, z_1) & \mathbb{S}(z_m, z_2) & \cdots & \mathbb{S}(z_m, z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

Proof. Consider the module map $X_{\mathcal{Z}, \mathcal{W}}$ on the quotient module $\mathcal{Q}_{\mathcal{Z}}$ as (see Lemma 6.3)

$$X_{\mathcal{Z}, \mathcal{W}}^* \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j),$$

for all $j = 1, \dots, m$. Define

$$\psi_{\mathcal{Z}, \mathcal{W}} = X_{\mathcal{Z}, \mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1). \quad (6.1)$$

We note the crucial fact that (as $\mathcal{Q}_{\mathcal{Z}} \subset H^\infty(\mathbb{D}^n)$, or see Corollary 6.5)

$$\psi_{\mathcal{Z}, \mathcal{W}} \in H^\infty(\mathbb{D}^n).$$

Claim: A function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ interpolates $\{z_i\}_{i=1}^m \subset \mathbb{D}^n$ and $\{w_i\}_{i=1}^m \subset \mathbb{D}$, that is

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if

$$S_\varphi = X_{\mathcal{Z}, \mathcal{W}}.$$

Indeed, since $\mathbb{S}(\cdot, z_i) \in \mathcal{Q}_{\mathcal{Z}}$, it follows that $P_{\mathcal{Q}_{\mathcal{Z}}} \mathbb{S}(\cdot, z_i) = \mathbb{S}(\cdot, z_i)$ and hence

$$\begin{aligned} S_\varphi^* \mathbb{S}(\cdot, z_i) &= P_{\mathcal{Q}_{\mathcal{Z}}} T_\varphi^* \mathbb{S}(\cdot, z_i) \\ &= \overline{\varphi(z_i)} \mathbb{S}(\cdot, z_i), \end{aligned}$$

for all $i = 1, \dots, m$. The definition of $X_{\mathcal{Z}, \mathcal{W}}$ now supports the claim. Of course, φ is a lift of $X_{\mathcal{Z}, \mathcal{W}}$. Then, by Theorem 4.1, it follows that $\varphi \in \mathcal{S}(\mathbb{D}^n)$ interpolates $\{z_i\}_{i=1}^m$ and $\{w_i\}_{i=1}^m$ if and only if

$$\mathfrak{I}_{\mathcal{Z}, \mathcal{W}} f = \int_{\mathbb{T}^n} \psi_{\mathcal{Z}, \mathcal{W}} f d\mu \quad (f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}),$$

defines a contraction $\mathfrak{I}_{\mathcal{Z}, \mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \rightarrow \mathbb{C}$. This proves the first half of the theorem. Now all that is left to do is calculate the representation of $\psi_{\mathcal{Z}, \mathcal{W}}$. Corollary 6.5 says that

$$X_{\mathcal{Z}, \mathcal{W}} = S_\varphi = S_{\psi_{\mathcal{Z}, \mathcal{W}}}.$$

Since $\psi_{\mathcal{Z}, \mathcal{W}} \in \mathcal{Q}_{\mathcal{Z}}$, there exists scalars $\{c_i\}_{i=1}^m$ such that

$$\psi_{\mathcal{Z}, \mathcal{W}} = \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i).$$

To compute the coefficients $\{c_i\}_{i=1}^m$ of the preceding expansion, we employ both reproducing kernel Hilbert space methods and conventional linear algebra. Fix $j \in \{1, \dots, m\}$. Then

$$\begin{aligned} X_{\mathcal{Z}, \mathcal{W}}^* \mathbb{S}(\cdot, z_j) &= S_{\psi_{\mathcal{Z}, \mathcal{W}}}^* \mathbb{S}(\cdot, z_j) \\ &= \overline{\psi_{\mathcal{Z}, \mathcal{W}}(z_j)} \mathbb{S}(\cdot, z_j), \end{aligned}$$

where, on the other hand, $X_{\mathcal{Z}, \mathcal{W}}^* \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j)$. Therefore

$$w_j = \psi_{\mathcal{Z}, \mathcal{W}}(z_j),$$

and hence, by the reproducing property of kernel functions (2.4), it follows that

$$\begin{aligned} w_j &= \psi_{\mathcal{Z}, \mathcal{W}}(z_j) \\ &= \left\langle \psi_{\mathcal{Z}, \mathcal{W}}, \mathbb{S}(\cdot, z_j) \right\rangle_{H^2(\mathbb{T}^n)} \\ &= \left\langle \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i), \mathbb{S}(\cdot, z_j) \right\rangle_{H^2(\mathbb{T}^n)} \\ &= \sum_{i=1}^m c_i \mathbb{S}(z_j, z_i), \end{aligned}$$

for all $j = 1, \dots, m$. In other words, we have

$$\begin{bmatrix} \mathbb{S}(z_1, z_1) & \mathbb{S}(z_1, z_2) & \cdots & \mathbb{S}(z_1, z_m) \\ \mathbb{S}(z_2, z_1) & \mathbb{S}(z_2, z_2) & \cdots & \mathbb{S}(z_2, z_m) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{S}(z_m, z_1) & \mathbb{S}(z_m, z_2) & \cdots & \mathbb{S}(z_m, z_m) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix},$$

equivalently

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \mathbb{S}(z_1, z_1) & \mathbb{S}(z_1, z_2) & \cdots & \mathbb{S}(z_1, z_m) \\ \mathbb{S}(z_2, z_1) & \mathbb{S}(z_2, z_2) & \cdots & \mathbb{S}(z_2, z_m) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{S}(z_m, z_1) & \mathbb{S}(z_m, z_2) & \cdots & \mathbb{S}(z_m, z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

The above $m \times m$ matrix is nothing but the Gram matrix of the linearly independent kernel functions $\{\mathbb{S}(\cdot, z_i) : i = 1, \dots, m\}$. The invertibility of the matrix is now immediate. \square

For solutions to the interpolation problem in the setting of bounded harmonic functions and H^p functions, we refer the reader to [31]

7. QUANTITATIVE INTERPOLATION AND EXAMPLES

This section is a continuation of our investigation into the interpolation problem. To begin, we will provide a quantitative solution to the interpolation problem on \mathbb{D}^n . The quantitative solution will then be employed to generate examples of interpolation with interpolating functions in $\mathcal{S}(\mathbb{D}^n)$, $n \geq 2$.

Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. As usual, define the m -dimensional zero-based quotient module $\mathcal{Q}_{\mathcal{Z}}$ of $H^2(\mathbb{T}^n)$ by

$$\mathcal{Q}_{\mathcal{Z}} = \text{span}\{\mathbb{S}(\cdot, z_i) : i = 1, \dots, m\}, \quad (7.1)$$

and $X_{\mathcal{Z}, \mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by

$$X_{\mathcal{Z}, \mathcal{W}}^* \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j) \quad (j = 1, \dots, m).$$

As observed in Lemma 6.3, $X_{\mathcal{Z}, \mathcal{W}}$ is a module map, and hence Corollary 6.5 implies

$$X_{\mathcal{Z}, \mathcal{W}} = S_{\psi}, \quad (7.2)$$

where

$$\psi := X_{\mathcal{Z}, \mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1).$$

Recall that

$$\psi \in \mathcal{Q}_{\mathcal{Z}} \subset H^\infty(\mathbb{D}^n).$$

On the other hand, as observed in the proof of Theorem 6.6 (more specifically, the claim part in the proof of Theorem 6.6), there exists a function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if

$$S_{\varphi} = X_{\mathcal{Z}, \mathcal{W}}.$$

Equivalently, $X_{\mathcal{Z}, \mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$ is a contraction and admits a lift (namely, $\varphi \in \mathcal{S}(\mathbb{D}^n)$). Based on Theorem 4.3 and the fact that $\psi = X_{\mathcal{Z}, \mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1)$, this is the same as saying that

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}\right) \geq 1,$$

where $\tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}} = (\mathcal{Q}_{\mathcal{Z}}^{\text{conj}} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n))$. This results in the quantitative solution to the interpolation problem:

Theorem 7.1. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. Suppose $\psi := X_{\mathcal{Z}, \mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1)$ and*

$$\tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}} = (\mathcal{Q}_{\mathcal{Z}}^{\text{conj}} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}\right) \geq 1.$$

Moreover, in this case, we have

$$\psi(z_i) = w_i,$$

for all $i = 1, \dots, m$.

Here is how the proof of the final assertion works: For each $i = 1, \dots, m$, in view of the definition of $X_{\mathcal{Z}, \mathcal{W}}$ and (7.2), we compute

$$\begin{aligned} \bar{w}_i \mathbb{S}(\cdot, z_i) &= X_{\mathcal{Z}, \mathcal{W}}^* \mathbb{S}(\cdot, z_i) \\ &= S_{\psi}^* \mathbb{S}(\cdot, z_i) \\ &= P_{\mathcal{Q}_{\mathcal{Z}}} T_{\psi}^* \mathbb{S}(\cdot, z_i) \\ &= \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i), \end{aligned}$$

as $\mathbb{S}(\cdot, z_i) \in \mathcal{Q}_{\mathcal{Z}}$. Therefore, $\psi(z_i) = w_i$ for all $i = 1, \dots, m$, which completes the proof. The final assertion will play an important role in the discussion that follows.

The rest of this section will be devoted to exploring examples of interpolation. We need to prove two lemmas. Before doing so, let us standardize some notations. We will set aside $m \geq 2$ as the number of nodes of the given interpolation data. We use bold letters such as \mathbf{a} , \mathbf{v} , \mathbf{w} , etc. to denote vectors in \mathbb{C}^m . For instance

$$\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{C}^m.$$

Also, denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^m}$ the standard inner product on \mathbb{C}^m . In particular

$$\|\mathbf{a}\|_{\mathbb{C}^m} = \left(\sum_{i=1}^m |a_i|^2 \right)^{\frac{1}{2}}.$$

We write

$$\mathbf{a}^{\perp} = \{\mathbf{v} \in \mathbb{C}^m : \langle \mathbf{a}, \mathbf{v} \rangle_{\mathbb{C}^m} = 0\}.$$

In view of the above notation, for each $\mathbf{a} \in \mathbb{C}^m$, we have the orthogonal decomposition

$$\mathbb{C}^m = \mathbb{C}\mathbf{a} \oplus \mathbf{a}^{\perp}.$$

We will work in the following general setting: Fix m distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ and m scalars $\{w_i\}_{i=1}^m \subset \mathbb{D}$. The quotient module of interest will be $\mathcal{Q}_{\mathcal{Z}} \subset H^{\infty}(\mathbb{D}^n)$ as defined in (7.1).

Lemma 7.2. *Let $\psi \in \mathcal{Q}_{\mathcal{Z}}$, and suppose $\psi(z_i) = w_i$ for all $i = 1, \dots, m$. Then there exists $\mathbf{v} \in \mathbf{w}^{\perp}$ such that*

$$\psi = \frac{\|\psi\|_2^2}{\|\mathbf{w}\|_{\mathbb{C}^m}^2} \sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i).$$

Proof. Since $\psi \in \mathcal{Q}_{\mathcal{Z}}$, there exists $\mathbf{c} \in \mathbb{C}^m$ such that

$$\psi = \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i).$$

Moreover, there exist a scalar $\alpha \in \mathbb{C}$ and a vector $\mathbf{v} \in \mathbf{c}^{\perp}$ such that $\mathbf{c} = \alpha \mathbf{w} \oplus \mathbf{v}$. Then

$$\psi = \alpha \sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i).$$

By assumption, $\psi \in H^\infty(\mathbb{D}^n)$ and $\psi(z_i) = w_i$ for all $i = 1, \dots, m$. The above equality then results in

$$\begin{aligned}
\|\psi\|_2^2 &= \left\langle \alpha \sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i), \psi \right\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \left\langle T_\psi^* \left(\sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i) \right), 1 \right\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \left\langle \left(\sum_{i=1}^m w_i \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i) \right), 1 \right\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \left\langle \left(\sum_{i=1}^m |w_i|^2 \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \bar{w}_i \mathbb{S}(\cdot, z_i) \right), 1 \right\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \|\mathbf{w}\|_{\mathbb{C}^m}^2 + \sum_{i=1}^m v_i \bar{w}_i \\
&= \alpha \|\mathbf{w}\|_{\mathbb{C}^m}^2,
\end{aligned}$$

as $\mathbf{v} \perp \mathbf{w}$. We have also used the general property that $\langle \mathbb{S}(\cdot, w), 1 \rangle_{H^2(\mathbb{T}^n)} = 1$ for all $w \in \mathbb{D}^n$. The above identity yields

$$\alpha = \frac{\|\psi\|_2^2}{\|\mathbf{w}\|_{\mathbb{C}^m}^2},$$

which completes the proof of the lemma. \square

The proof of the following lemma is similar to the proof of the previous one.

Lemma 7.3. *Let $\psi \in \mathcal{Q}_{\mathcal{Z}}$, and suppose $\psi(z_i) = w_i$ for all $i = 1, \dots, m$. Then*

$$\mathcal{Q}_{\mathcal{Z}} \ominus \{\psi\} = \left\{ \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i) : \mathbf{v} \in \mathbf{w}^\perp \right\}.$$

Proof. Given $\mathbf{v} \in \mathbb{C}^m$, observe that

$$\sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i) \perp \psi,$$

if and only if

$$\begin{aligned}
0 &= \left\langle \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i), \psi \right\rangle_{H^2(\mathbb{T}^n)} \\
&= \left\langle T_\psi^* \left(\sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i) \right), 1 \right\rangle_{H^2(\mathbb{T}^n)} \\
&= \left\langle \sum_{i=1}^m v_i \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i), 1 \right\rangle_{H^2(\mathbb{T}^n)} \\
&= \sum_{i=1}^m v_i \bar{w}_i \\
&= \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{C}^m}.
\end{aligned}$$

This completes the proof of the lemma. \square

Now we are ready for examples of interpolation on \mathbb{D}^n , $n \geq 2$. First, we elaborate on the construction of the 3-point interpolation problem. Suppose:

- (1) $\{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2\}$ is an orthogonal basis for \mathbb{C}^3 , where

$$\begin{cases} \mathbf{b}_0 = (1, 1, 1) \\ \mathbf{b}_1 = (\zeta_{11}, \zeta_{12}, \zeta_{13}) \\ \mathbf{b}_2 = (\zeta_{21}, \zeta_{22}, \zeta_{23}). \end{cases}$$

- (2) $\|\mathbf{b}_1\|_{\mathbb{C}^3}, \|\mathbf{b}_2\|_{\mathbb{C}^3} \geq 1$.

- (3) $\{z_1, z_2, z_3\} \subset \mathbb{D}^n$ are three distinct points such that

$$\begin{cases} z_1 = (\zeta_{11}, \zeta_{21}, \tilde{z}_1) \\ z_2 = (\zeta_{12}, \zeta_{22}, \tilde{z}_2) \\ z_3 = (\zeta_{13}, \zeta_{23}, \tilde{z}_3), \end{cases}$$

for some (arbitrary) $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \mathbb{D}^{n-2}$.

- (4) $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{D}^3$ such that $\|\mathbf{w}\|_{\mathbb{C}^3} \leq \frac{1}{\sqrt{3}}$.

Claim: There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, 2, 3$.

Here is how the proof of the claim goes: In view of Theorem 7.1, it is enough to prove that

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_Z}\right) \geq 1,$$

where $\psi = X_{Z, \mathcal{W}}(P_{\mathcal{Q}_Z} 1)$ and

$$\tilde{\mathcal{M}}_{\mathcal{Q}_Z} = (\mathcal{Q}_Z^{\text{conj}} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Recall that $X_{Z, \mathcal{W}} \in \mathcal{B}(\mathcal{Q}_Z)$ is defined by

$$X_{Z, \mathcal{W}}^* \mathbb{S}(\cdot, z_i) = \bar{w}_i \mathbb{S}(\cdot, z_i),$$

for all $i = 1, \dots, m$. Also recall the crucial fact that (see Theorem 7.1)

$$\psi(z_i) = w_i \quad (i = 1, \dots, m).$$

Using the conjugation invariance property of L^1 -norm (that is, $\|f\|_{L^1(\mathbb{T}^n)} = \|\bar{f}\|_{L^1(\mathbb{T}^n)}$ for all $f \in L^1(\mathbb{T}^n)$), we infer that

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_Z}\right) = \text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\psi}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_Z}^{\text{conj}}\right),$$

where (recall that $\mathcal{M}_n^{\text{conj}} = \mathcal{M}_n$)

$$\tilde{\mathcal{M}}_{\mathcal{Q}_Z}^{\text{conj}} = (\mathcal{Q}_Z \ominus \{\psi\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)^{\text{conj}}).$$

It will be convenient (as well as enough) to prove that

$$\text{dist}_{L^1(\mathbb{T}^n)}\left(\frac{\psi}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_Z}^{\text{conj}}\right) \geq 1.$$

Also, to avoid notational confusion, we use $\{Z_1, \dots, Z_n\}$ for the variables of \mathbb{C}^n . By the definition of Szegő kernel, we have

$$\begin{cases} \mathbb{S}(\cdot, z_1) = 1 + \bar{\zeta}_{11}Z_1 + \bar{\zeta}_{21}Z_2 + \dots \\ \mathbb{S}(\cdot, z_2) = 1 + \bar{\zeta}_{12}Z_1 + \bar{\zeta}_{22}Z_2 + \dots \\ \mathbb{S}(\cdot, z_3) = 1 + \bar{\zeta}_{13}Z_1 + \bar{\zeta}_{23}Z_2 + \dots \end{cases} \quad (7.3)$$

We will need to prove the following inequality

$$\left\| \frac{\psi}{\|\psi\|_2^2} + f \right\|_{L^1(\mathbb{T}^n)} \geq 1 \quad (f \in \tilde{\mathcal{M}}_{\mathcal{Q}_Z}^{conj}).$$

Since $\psi(z_i) = w_i$ for all $i = 1, \dots, m$, in view of Lemma 7.3, an element $f \in \tilde{\mathcal{M}}_{\mathcal{Q}_Z}^{conj}$ admits the following representation

$$f = \sum_{i=1}^3 v_i \mathbb{S}(\cdot, z_i) + \tilde{f},$$

for some $\mathbf{v} \in \mathbf{w}^\perp$ and $\tilde{f} \in \mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)^{conj}$. Therefore, for each $f \in \tilde{\mathcal{M}}_{\mathcal{Q}_Z}^{conj}$, by Lemma 7.2, we conclude that

$$\frac{\psi}{\|\psi\|_2^2} + f = \sum_{i=1}^3 \frac{w_i}{\|\mathbf{w}\|_{\mathbb{C}^3}^2} \mathbb{S}(\cdot, z_i) + \sum_{i=1}^3 v_i \mathbb{S}(\cdot, z_i) + \tilde{f},$$

for some $\mathbf{v} \in \mathbf{w}^\perp$ and $\tilde{f} \in \mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)^{conj}$. For each $\mathbf{v} \in \mathbf{w}^\perp$, we set

$$F_{\mathbf{v}} = \frac{1}{\|\mathbf{w}\|_{\mathbb{C}^3}^2} \mathbf{w} \oplus \mathbf{v}.$$

It is important to keep in mind that \mathbf{v} and \tilde{f} depend on f . By assumption, $\|\mathbf{w}\|_{\mathbb{C}^3} \leq \frac{1}{\sqrt{3}}$, and hence

$$\|F_{\mathbf{v}}\|_{\mathbb{C}^3} \geq \sqrt{3}.$$

Using the kernel functions' power series expansion as in (7.3), we find

$$\begin{aligned} \frac{\psi}{\|\psi\|_2^2} + f &= \sum_{i=1}^3 \frac{w_i}{\|\mathbf{w}\|_{\mathbb{C}^3}^2} \mathbb{S}(\cdot, z_i) + \sum_{i=1}^3 v_i \mathbb{S}(\cdot, z_i) + \tilde{f} \\ &= \left(\langle F_{\mathbf{v}}, \mathbf{b}_0 \rangle_{\mathbb{C}^3} 1 + \langle F_{\mathbf{v}}, \mathbf{b}_1 \rangle_{\mathbb{C}^3} Z_1 + \langle F_{\mathbf{v}}, \mathbf{b}_2 \rangle_{\mathbb{C}^3} Z_2 + \dots \right) + \tilde{f}. \end{aligned}$$

There exists $i \in \{0, 1, 2\}$ such that

$$|\langle F_{\mathbf{v}}, \mathbf{b}_i \rangle_{\mathbb{C}^3}| \geq 1.$$

If not, suppose $|\langle F_{\mathbf{v}}, \mathbf{b}_i \rangle_{\mathbb{C}^3}| < 1$ for all $i = 0, 1, 2$. Then

$$\left| \left\langle F_{\mathbf{v}}, \|\mathbf{b}_i\|_{\mathbb{C}^3} \left(\frac{1}{\|\mathbf{b}_i\|_{\mathbb{C}^3}} \mathbf{b}_i \right) \right\rangle \right| < 1,$$

implies

$$\left| \left\langle F_{\mathbf{v}}, \left(\frac{1}{\|\mathbf{b}_i\|_{\mathbb{C}^3}} \mathbf{b}_i \right) \right\rangle \right| < \frac{1}{\|\mathbf{b}_i\|_{\mathbb{C}^3}} \leq 1,$$

for all $i = 0, 1, 2$. Since

$$\left\{ \frac{1}{\|\mathbf{b}_i\|_{\mathbb{C}^3}} \mathbf{b}_i \right\}_{i=0}^2,$$

is an orthonormal basis for \mathbb{C}^3 , the above inequality contradicts the fact that $\|F_{\mathbf{v}}\|_{\mathbb{C}^3} \geq \sqrt{3}$. On the other hand, since \tilde{f} does not have an analytic part and

$$\langle \tilde{f}, 1 \rangle_{L^2(\mathbb{T}^n)} = 0,$$

it follows that

$$\left\langle \frac{\psi}{\|\psi\|_2^2} + f, g \right\rangle_{L^2(\mathbb{T}^n)} = \begin{cases} \langle F_{\mathbf{v}}, \mathbf{b}_0 \rangle_{\mathbb{C}^3} & \text{if } g = 1 \\ \langle F_{\mathbf{v}}, \mathbf{b}_1 \rangle_{\mathbb{C}^3} & \text{if } g = Z_1 \\ \langle F_{\mathbf{v}}, \mathbf{b}_2 \rangle_{\mathbb{C}^3} & \text{if } g = Z_2. \end{cases}$$

Therefore

$$\left| \left\langle \frac{\psi}{\|\psi\|_2^2} + f, g \right\rangle_{L^2(\mathbb{T}^n)} \right| \geq 1,$$

for some $g \in \{1, Z_1, Z_2\}$. On the other hand (see the duality (4.1))

$$\left\langle \frac{\psi}{\|\psi\|_2^2} + f, g \right\rangle_{L^2(\mathbb{T}^n)} = \begin{cases} \chi_1 \left(\frac{\psi}{\|\psi\|_2^2} + f \right) & \text{if } g = 1 \\ \chi_{\bar{Z}_1} \left(\frac{\psi}{\|\psi\|_2^2} + f \right) & \text{if } g = Z_1 \\ \chi_{\bar{Z}_2} \left(\frac{\psi}{\|\psi\|_2^2} + f \right) & \text{if } g = Z_2. \end{cases}$$

However

$$\|\chi_{\bar{g}}\| = 1,$$

for all $g \in \{1, Z_1, Z_2\}$, and hence

$$\left\| \frac{\psi}{\|\psi\|_2^2} + f \right\|_{L^1(\mathbb{T}^n)} \geq 1,$$

for all $f \in \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}^{\text{conj}}$. This completes the proof of the claim. Furthermore, in this case, we can specify an explicit interpolating function. Note that $\{\mathbf{e}_i\}_{i=0}^2$ is an orthonormal basis for \mathbb{C}^3 , where

$$\mathbf{e}_i = \frac{1}{\|\mathbf{b}_i\|_{\mathbb{C}^3}} \mathbf{b}_i,$$

for all $i = 0, 1, 2$. We write

$$\mathbf{w} = \sum_{i=0}^2 \alpha_i \mathbf{e}_i,$$

and set

$$\varphi(Z) = \frac{\alpha_0}{\|\mathbf{b}_0\|_{\mathbb{C}^3}} + \frac{\alpha_1}{\|\mathbf{b}_1\|_{\mathbb{C}^3}} Z_1 + \frac{\alpha_2}{\|\mathbf{b}_2\|_{\mathbb{C}^3}} Z_2,$$

for all $Z = (Z_1, \dots, Z_n) \in \mathbb{D}^n$. Since $\|\mathbf{w}\|_{\mathbb{C}^3}^2 \leq \frac{1}{3}$, it follows that

$$\sum_{i=0}^2 |\alpha_i|^2 \leq \frac{1}{3},$$

and hence, by the Cauchy-Schwarz inequality, we conclude that

$$\sum_{i=0}^2 |\alpha_i| \leq 1.$$

Moreover, since $\|\mathbf{b}_i\|_{\mathbb{C}^3} \geq 1$ for all $i = 0, 1, 2$, for each $Z \in \mathbb{D}^n$, we infer that

$$\begin{aligned} |\varphi(Z)| &= \left| \frac{\alpha_0}{\|\mathbf{b}_0\|_{\mathbb{C}^3}} + \frac{\alpha_1}{\|\mathbf{b}_1\|_{\mathbb{C}^3}} Z_1 + \frac{\alpha_2}{\|\mathbf{b}_2\|_{\mathbb{C}^3}} Z_2 \right| \\ &\leq \frac{|\alpha_0|}{\|\mathbf{b}_0\|_{\mathbb{C}^3}} + \frac{|\alpha_1|}{\|\mathbf{b}_1\|_{\mathbb{C}^3}} |Z_1| + \frac{|\alpha_2|}{\|\mathbf{b}_2\|_{\mathbb{C}^3}} |Z_2| \\ &\leq \sum_{i=0}^2 |\alpha_i| \\ &\leq 1, \end{aligned}$$

and consequently, $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Finally, we compute

$$\begin{aligned} \varphi(z_i) &= \frac{\alpha_0}{\|\mathbf{b}_0\|_{\mathbb{C}^3}} + \frac{\alpha_1}{\|\mathbf{b}_1\|_{\mathbb{C}^3}} \zeta_{1i} + \frac{\alpha_2}{\|\mathbf{b}_2\|_{\mathbb{C}^3}} \zeta_{2i} \\ &= \alpha_0 \pi_i(\mathbf{e}_0) + \alpha_1 \pi_i(\mathbf{e}_1) + \alpha_2 \pi_i(\mathbf{e}_2) \\ &= \pi_i\left(\sum_{j=0}^2 \alpha_j \mathbf{e}_j\right) \\ &= \pi_i(\mathbf{w}) \\ &= w_i, \end{aligned}$$

for all $i = 1, 2, 3$. Therefore, φ is a solution to the interpolation problem with data $\{z_i\}_{i=1}^3 \subset \mathbb{D}^n$ and $\{w_i\}_{i=1}^3 \subset \mathbb{D}$.

For general m -point interpolation, $m \geq 2$, the same proof concept applies, but the computation would be more laborious. We only report the general result and leave the other details to the interested readers.

Theorem 7.4. *Let $n \geq 2$, $m \geq 3$, and suppose $n \geq m - 1$. Let $\{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. Suppose $\{\mathbf{b}_i\}_{i=0}^{m-1} \subset \mathbb{C}^m$, where $\mathbf{b}_0 = (1, \dots, 1)$, and*

$$\mathbf{b}_j = (\zeta_{j1}, \zeta_{j2}, \dots, \zeta_{jm}),$$

for all $j = 1, \dots, m - 1$. Assume that:

- (1) $\{\mathbf{b}_i\}_{i=0}^{m-1}$ is an orthogonal basis for \mathbb{C}^m .
- (2) $\|\mathbf{b}_i\|_{\mathbb{C}^m} \geq 1$ for all $i = 1, \dots, m - 1$.
- (3) $z_j = (\zeta_{1j}, \zeta_{2j}, \dots, \zeta_{m-1,j}, \tilde{z}_j)$, where $\tilde{z}_j \in \mathbb{D}^{n-m+1}$ arbitrary, and $j = 1, \dots, m$.
- (4) $\|\mathbf{w}\|_{\mathbb{C}^m} \leq \frac{1}{\sqrt{n}}$, where $\mathbf{w} = (w_1, \dots, w_n)$.

Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$. Furthermore, φ can be chosen as a polynomial.

Evidently, there is no dearth of examples of data that meet the aforementioned conditions. The following remark elaborates on this:

Remark 7.5. If the number of variables $n (\geq 2)$ and the number of nodes $m (\geq 3)$ satisfies the condition $n \geq m - 1$, and if one restricts the first $m - 1$ slots of the coordinates of the

interpolation nodes $\{z_i\}_{i=1}^m$ (so that the corresponding columns along with the constant vector 1 forms a basis of \mathbb{C}^m) along with the norm bound on \mathbf{w} as

$$\|\mathbf{w}\|_{\mathbb{C}^m} \leq \frac{1}{n},$$

then one can ensure that interpolation will occur for any choice of $\{\tilde{z}_i\}_{i=1}^m \subset \mathbb{D}^{n-m+1}$. The relationship between the orthogonal set of vectors $\{\mathbf{b}_i\}_{i=1}^{m-1} \subset \mathbb{C}^m$ and interpolation nodes $\{z_i\}_{i=1}^m \subset \mathbb{D}^n$ can be represented by the formal matrix:

$$\begin{matrix} & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \cdots & \mathbf{b}_{m-1} \\ \begin{matrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{matrix} & \begin{pmatrix} \zeta_{11} & \zeta_{21} & \zeta_{31} & \cdots & \zeta_{m-1,1} & \cdots \\ \zeta_{12} & \zeta_{22} & \zeta_{32} & \cdots & \zeta_{m-1,2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \zeta_{1m} & \zeta_{2m} & \zeta_{3m} & \cdots & \zeta_{m-1,m} & \cdots \end{pmatrix} \end{matrix}.$$

What this means is that there is an abundance of examples of interpolation in hand in several variables.

We refer the reader to [47] for interpolation from operator algebraic perspective.

8. COMMUTANT LIFTING AND EXAMPLES

This section contains illustrations of commutant lifting on quotient modules of $H^2(\mathbb{T}^n)$, $n > 1$. Our first aim is to validate the examples in Section 3 using our commutant lifting theorem. We begin with a lemma.

Lemma 8.1. *Let $h \in H^2(\mathbb{T}^n)$. Then $\|h\|_1 = \|h\|_2 = 1$ if and only if h is inner.*

Proof. Suppose $\|h\|_1 = \|h\|_2 = 1$. In particular, $h \in H^1(\mathbb{T}^n) \subseteq L^1(\mathbb{T}^n)$. By the Hahn–Banach theorem, there exists $\varphi \in L^\infty(\mathbb{T}^n)$ such that $\|\varphi\|_\infty = 1$ (as $\|h\|_1 = 1$) and

$$\int_{\mathbb{T}^n} h\varphi d\mu = \|h\|_1 = 1.$$

In the above, we used the duality $(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n)$ once more. We claim that φ is unimodular. Indeed, if

$$|\varphi| < 1 \text{ on } A,$$

for some measurable set $A \subseteq \mathbb{T}^n$ such that $\mu(A) > 0$, then

$$\begin{aligned} 1 &= \left| \int_{\mathbb{T}^n} h\varphi d\mu \right| \\ &\leq \left| \int_{A^c} h\varphi d\mu \right| + \left| \int_A h\varphi d\mu \right| \\ &\leq \int_{A^c} |h||\varphi| d\mu + \int_A |h||\varphi| d\mu \\ &< \int_{A^c} |h| d\mu + \int_A |h| d\mu \\ &= \|h\|_1, \end{aligned}$$

that is, $1 < \|h\|_1$, a contradiction. Since $h \in H^2(\mathbb{D}^n) \subseteq L^2(\mathbb{T}^n)$, we find a scalar c and a function $g \in L^2(\mathbb{T}^n)$ such that

$$\varphi = c\bar{h} \oplus g.$$

Observe that $\langle \bar{h}, g \rangle = \langle h, \bar{g} \rangle = 0$. Therefore

$$\begin{aligned} 1 &= \int_{\mathbb{T}^n} h\varphi d\mu \\ &= \int_{\mathbb{T}^n} h(c\bar{h} \oplus g) d\mu \\ &= \langle h, c\bar{h} \oplus \bar{g} \rangle_{L^2(\mathbb{T}^n)} \\ &= c, \end{aligned}$$

and hence, $\varphi = \bar{h} \oplus g$. Then

$$\begin{aligned} 1 + \|g\|_2^2 &= \|h\|_2^2 + \|g\|_2^2 \\ &= \|\varphi\|_2^2 \\ &\leq \|\varphi\|_\infty^2 \\ &= 1, \end{aligned}$$

implies that $g = 0$, and hence $\varphi = \bar{h}$. Since $\varphi \in L^\infty(\mathbb{T}^n)$, it follows that $h \in H^\infty(\mathbb{D}^n)$ is an inner function. The converse simply follows from the integral representation of norms on $H^2(\mathbb{T}^n)$ and $H^1(\mathbb{T}^n)$ and the fact that $|h| = 1$ a.e. on \mathbb{T}^n . \square

Now we follow the setting of Corollary 3.3: For a fixed $m \in \mathbb{N}$, we consider the homogeneous quotient module

$$\mathcal{Q}_m = \bigoplus_{t=0}^m H_t,$$

a homogeneous polynomial $p \in \mathcal{Q}_m$ as

$$p = \sum_{|k|=m} a_k z^k,$$

with $\|p\|_2 = 1$, and that $a_k, a_l \neq 0$ for some $k \neq l$ in \mathbb{Z}_+^n . We know, by Theorem 4.1, that S_p is liftable if and only if

$$X_{\mathcal{Q}_m}(f) = \int_{\mathbb{T}^n} \psi f d\mu \quad (f \in \mathcal{M}_{\mathcal{Q}_m}),$$

defines a contraction on $(\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1)$, where $\psi = S_p(P_{\mathcal{Q}_m}1)$. Since 1 and p are in \mathcal{Q}_m , it follows that $\psi = p$, and hence

$$\begin{aligned} X_{\mathcal{Q}_m}(\bar{p}) &= \int_{\mathbb{T}^n} \bar{p}\psi d\mu \\ &= \int_{\mathbb{T}^n} |p|^2 d\mu \\ &= 1. \end{aligned}$$

However

$$\|\bar{p}\|_1 = \|p\|_1 < 1.$$

Indeed, since $a_k, a_l \neq 0$, Lemma 3.1 ensures that p is not inner. This, together with the fact that $\|p\|_2 = 1$ and Lemma 8.1 completes the proof of the claim. Therefore, $X_{\mathcal{Q}_m}$ on $(\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1)$ is not a contraction, and hence S_p is not liftable. As a result, we recover Corollary 3.3 using Theorem 4.1.

The idea used in the preceding example can be extended to provide further nontrivial examples of module maps that do not admit any lift. The following is an example, and this time we will use Theorem 4.1 directly to prove that such a module map does not lift. Suppose $n > 1$. Consider the submodule

$$\mathcal{S} = z_1 \cdots z_n H^2(\mathbb{T}^n).$$

We will be working on the corresponding quotient module $\mathcal{Q} = \mathcal{S}^\perp$. Clearly

$$\mathcal{Q} = \ker \left(\prod_{i=1}^n T_{z_i}^* \right).$$

We observe that

$$\mathcal{Q} = H_{z_1}^2(\mathbb{T}^n) + \cdots + H_{z_n}^2(\mathbb{T}^n),$$

where $H_{z_i}^2(\mathbb{T}^n)$, $i = 1, \dots, n$, is the closed subspace of $H^2(\mathbb{T}^n)$ of functions that are independent of the z_i variable, or equivalently

$$H_{z_i}^2(\mathbb{T}^n) = \ker T_{z_i}^*.$$

Indeed, it is clear that $H_{z_1}^2(\mathbb{T}^n) + \cdots + H_{z_n}^2(\mathbb{T}^n) \subseteq \mathcal{Q}$. Let $f \in \ker(\prod_{i=1}^n T_{z_i}^*)$, and suppose

$$f = \sum_{k \in \mathbb{Z}_+^n} a_k z^k.$$

Then

$$\begin{aligned} 0 &= T_{z_1 \cdots z_n}^* f \\ &= P_{H^2(\mathbb{T}^n)} \left(\sum_{k \in \mathbb{Z}_+^n} a_k \bar{z}_1 \cdots \bar{z}_n z^k \right), \end{aligned}$$

implies

$$\sum_{k \in \mathbb{Z}_+^n} a_k \bar{z}_1 \cdots \bar{z}_n z^k \in (H^2(\mathbb{T}^n))^\perp.$$

In other words, if $a_k \neq 0$ for some $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, then we must have $k_i = 0$ for some $i = 1, \dots, n$. Therefore, there exists $f_i \in H_{z_i}^2(\mathbb{T}^n)$, $i = 1, \dots, n$, such that $f = f_1 + \cdots + f_n$. This proves the claim. Now for each $i = 1, \dots, n$, set

$$\zeta_i := \prod_{j \neq i} z_j,$$

and pick inner function $\varphi_i \in H_{z_i}^2(\mathbb{T}^n)$. Let $z_0 \in \mathbb{T}^n$ and suppose $\varphi_i(z_0)$ is well defined and

$$|\varphi_i(z_0)| = 1,$$

for all $i = 1, \dots, n$. Choose $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}_{\geq 0}$ such that $\alpha_p, \alpha_q \neq 0$ for some $p \neq q$, and

$$\sum_{i=1}^n \alpha_i^2 = 1.$$

The preceding set of assumptions ensures that

$$\sum_{i=1}^n \alpha_i > 1.$$

Finally, define $\varphi \in \mathcal{S}(\mathbb{D}^n)$ by

$$\varphi = \sum_{i=1}^n \alpha_i \bar{\beta}_i \zeta_i \varphi_i,$$

where $\beta_i = (\zeta_i \varphi_i)(z_0)$ for all $i = 1, \dots, n$. We claim that φ is not inner. Indeed, since

$$\varphi(z_0) = \sum_{i=1}^n \alpha_i \bar{\beta}_i \beta_i,$$

and $|\beta_i| = 1$, it follows that

$$\varphi(z_0) = \sum_{i=1}^n \alpha_i > 1.$$

Therefore, there exists $r \in (0, 1)$ such that (note that φ is well defined at z_0)

$$|\varphi(rz_0)| > 1,$$

and hence, by the maximum modulus theorem, we conclude that $\|\varphi\|_\infty > 1$, which completes the proof of the claim. Next, we claim that S_φ is a contraction. Fix $f \in \mathcal{Q}$. For each $i \in \{1, \dots, n\}$, we have

$$\mathcal{Q} \ominus H_{z_i}^2(\mathbb{T}^n) = (\ker T_{z_i}^*)^\perp \cap \mathcal{Q} = \{z_i g \in \mathcal{Q} : g \in H^2(\mathbb{T}^n)\},$$

and hence there exist $f_i \in H_{z_i}^2(\mathbb{T}^n)$ and $g_i \in H^2(\mathbb{T}^n)$ such that

$$f = f_i \oplus z_i g_i \in H_{z_i}^2(\mathbb{T}^n) \oplus (\mathcal{Q} \ominus H_{z_i}^2(\mathbb{T}^n)).$$

Then

$$\begin{aligned} S_{\zeta_i \varphi_i} f &= S_{\zeta_i \varphi_i} (f_i + z_i g_i) \\ &= S_{\zeta_i \varphi_i} f_i + P_{\mathcal{Q}}(\zeta_i \varphi_i z_i g_i) \\ &= S_{\zeta_i \varphi_i} f_i + P_{\mathcal{Q}}(z_1 \cdots z_n \varphi_i g_i) \\ &= S_{\zeta_i \varphi_i} f_i, \end{aligned}$$

as $z_1 \cdots z_n \varphi_i g_i \in \mathcal{S}$, and hence

$$S_{\zeta_i \varphi_i} f = S_{\zeta_i \varphi_i} P_{H_{z_i}^2(\mathbb{T}^n)} f.$$

Observe moreover that $\zeta_i \varphi_i H_{z_i}^2(\mathbb{T}^n) \subseteq H_{z_i}^2(\mathbb{T}^n)$. In view of $H_{z_i}^2(\mathbb{T}^n) \subseteq \mathcal{Q}$, we conclude that $S_{\zeta_i \varphi_i} P_{H_{z_i}^2(\mathbb{T}^n)} f = \zeta_i \varphi_i P_{H_{z_i}^2(\mathbb{T}^n)} f$, which yields

$$S_{\zeta_i \varphi_i} f = \zeta_i \varphi_i P_{H_{z_i}^2(\mathbb{T}^n)} f.$$

Therefore, for $i \neq j$, we have

$$\begin{aligned} \langle S_{\zeta_i \varphi_i} f, S_{\zeta_j \varphi_j} f \rangle &= \langle \zeta_i \varphi_i P_{H_{z_i}^2(\mathbb{T}^n)} f, \zeta_j \varphi_j P_{H_{z_j}^2(\mathbb{T}^n)} f \rangle \\ &= \langle T_{\zeta_j}^* \zeta_i \varphi_i P_{H_{z_i}^2(\mathbb{T}^n)} f, \varphi_j P_{H_{z_j}^2(\mathbb{T}^n)} f \rangle \\ &= \langle T_{z_i}^* z_j \varphi_i P_{H_{z_i}^2(\mathbb{T}^n)} f, \varphi_j P_{H_{z_j}^2(\mathbb{T}^n)} f \rangle \\ &= 0, \end{aligned}$$

as $z_j \varphi_i P_{H_{z_i}^2(\mathbb{T}^n)} f \in \ker T_{z_i}^*$. So we find

$$S_{\zeta_i \varphi_i} f \perp S_{\zeta_j \varphi_j} f \quad (i \neq j). \quad (8.1)$$

This allows us to compute the norm of $S_\varphi f$ as follows (note that $|\beta_i| = 1$ and $S_{\zeta_i \varphi_i}$ is a contraction for all $i = 1, \dots, n$):

$$\begin{aligned} \|S_\varphi f\|^2 &= \left\| \sum_{i=1}^n \alpha_i \bar{\beta}_i S_{\zeta_i \varphi_i} f \right\|^2 \\ &= \sum_{i=1}^n \alpha_i^2 \|S_{\zeta_i \varphi_i} f\|^2 \\ &\leq \sum_{i=1}^n \alpha_i^2 \|f\|^2 \\ &= \|f\|^2. \end{aligned}$$

This means that S_φ is a contraction. Our final claim is that S_φ is incapable of admitting any lift, which, in view of Theorem 4.1, is equivalent to the assertion that $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \rightarrow \mathbb{C}$ is not a contraction, where

$$X_\mathcal{Q} f = \int_{\mathbb{T}^n} \psi f d\mu \quad (f \in \mathcal{M}_\mathcal{Q}),$$

and

$$\psi = S_\varphi(P_\mathcal{Q} 1).$$

Indeed, since $1, \varphi \in \mathcal{Q}$, it follows that

$$\psi = \varphi.$$

On the other hand, since $\bar{\varphi} \in \mathcal{M}_\mathcal{Q}$ (recall that $\mathcal{M}_\mathcal{Q} = \mathcal{Q}^{conj} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n))$), we observe that

$$\begin{aligned} X_\mathcal{Q} \bar{\varphi} &= \int_{\mathbb{T}^n} \varphi \bar{\varphi} d\mu \\ &= \|\varphi\|_{H^2(\mathbb{T}^n)}^2 \\ &= 1. \end{aligned}$$

Finally, applying (8.1) to $f = 1 \in \mathcal{Q}$, we obtain that

$$\|\varphi\|_{H^2(\mathbb{T}^n)}^2 = 1.$$

This also follows from the equalities following (8.1) corresponding to the choice $f = 1$ along with the fact that $\zeta_i \varphi_i$ is inner for all $i = 1, \dots, n$. Since φ is not an inner function, by Lemma 8.1, we conclude that

$$\|\bar{\varphi}\|_1 = \|\varphi\|_1 < 1,$$

and hence $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \rightarrow \mathbb{C}$ is not a contraction. This proves the following result:

Proposition 8.2. *Let $\{\alpha_1, \dots, \alpha_n\} \subset \mathbb{R}_{\geq 0}$, suppose $\alpha_p, \alpha_q \neq 0$ for some $p \neq q$, and*

$$\sum_{i=1}^n \alpha_i^2 = 1.$$

Let $z_0 \in \mathbb{T}^n$, and let φ_i be an inner function independent of the variable z_i , and suppose $\varphi_i(z_0)$ is well defined and

$$|\varphi_i(z_0)| = 1,$$

for all $i = 1, \dots, n$. Define $\varphi \in \mathcal{S}(\mathbb{D}^n)$ by

$$\varphi = \sum_{i=1}^n \alpha_i \bar{\beta}_i \zeta_i \varphi_i,$$

where $\beta_i = (\zeta_i \varphi_i)(z_0)$ and $\zeta_i := \prod_{j \neq i} z_j$ for all $i = 1, \dots, n$. Then S_φ on $\mathcal{Q} = \ker(\prod_{i=1}^n T_{z_i}^*)$ does not admit any lift.

9. RECOVERING SARASON'S LIFTING THEOREM

In this section, we explain how to recover Sarason's commutant lifting theorem from Theorem 4.1. We will employ several tools (just like Sarason) that are commonly used and are valid only in one variable function theory. We start with the Beurling theorem [15]. Let $\mathcal{Q} \subsetneq H^2(\mathbb{T})$ be a closed subspace. Then \mathcal{Q} is a quotient module if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that $\mathcal{Q} = \mathcal{Q}_\theta$, where

$$\mathcal{Q}_\theta := H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T}).$$

Observe that $\theta H^2(\mathbb{T})$ is a closed subspace (as T_θ is an isometry on $H^2(\mathbb{T})$) and

$$\mathcal{Q}_\theta \cong H^2(\mathbb{T}) / \theta H^2(\mathbb{T}).$$

Therefore, quotient modules of $H^2(\mathbb{T})$ are inner function based - a typical one variable phenomenon (see Rudin [56] for counterexamples in several variables). In the following, we prove a key result.

Lemma 9.1. *Let $\theta \in H^\infty(\mathbb{D})$ be an inner function. Then*

$$\mathcal{Q}_\theta^{\text{conj}} \oplus zH^2(\mathbb{T}) = \bar{\theta}(zH^2(\mathbb{T})).$$

Proof. Let $g \in \mathcal{Q}_\theta$. Then $\bar{g} \in \mathcal{Q}_\theta^{\text{conj}}$, and hence, for each $m \geq 0$, we have

$$\begin{aligned} \langle \theta \bar{g}, \bar{z}^m \rangle_{L^2(\mathbb{T})} &= \overline{\langle \bar{\theta} g, z^m \rangle_{L^2(\mathbb{T})}} \\ &= \overline{\langle g, \theta z^m \rangle_{H^2(\mathbb{T})}} \\ &= 0, \end{aligned}$$

as $\theta z^m \in \mathcal{Q}_\theta^\perp$. This implies $\theta \mathcal{Q}_\theta^{\text{conj}} \subseteq zH^2(\mathbb{T})$ and hence $\mathcal{Q}_\theta^{\text{conj}} \subseteq \bar{\theta}(zH^2(\mathbb{T}))$. Also, for all $h \in H^2(\mathbb{T})$, since

$$zh = \bar{\theta}(\theta zh) = \bar{\theta}(z\theta h),$$

it follows that $zH^2(\mathbb{T}) \subseteq \bar{\theta}(zH^2(\mathbb{T}))$. Therefore

$$\mathcal{Q}_\theta^{\text{conj}} \oplus zH^2(\mathbb{T}) \subseteq \bar{\theta}(zH^2(\mathbb{T})).$$

For the reverse inclusion, first we observe that for $f \in \mathcal{Q}_\theta$ and $m \geq 1$, since

$$\begin{aligned} \langle \bar{\theta} z f, z^m \rangle_{L^2(\mathbb{T})} &= \langle z f, z \theta z^{m-1} \rangle_{H^2(\mathbb{T})} \\ &= \langle f, \theta z^{m-1} \rangle_{H^2(\mathbb{T})} \\ &= 0, \end{aligned}$$

it follows that $\bar{\theta}z\mathcal{Q}_\theta \perp zH^2(\mathbb{T})$, and hence $\bar{\theta}z\mathcal{Q}_\theta \subseteq H^2(\mathbb{T})^{\text{conj}}$. On the other hand, we know

$$H^2(\mathbb{T})^{\text{conj}} = \mathcal{Q}_\theta^{\text{conj}} \oplus (\theta H^2(\mathbb{T}))^{\text{conj}}.$$

In view of this, for each $f \in \mathcal{Q}_\theta$ and $g \in H^2(\mathbb{T})$, we further compute

$$\begin{aligned} \langle \bar{\theta}zf, \bar{\theta}\bar{g} \rangle_{L^2(\mathbb{T})} &= \langle zf, \bar{g} \rangle_{L^2(\mathbb{T})} \\ &= \langle f, \bar{z}\bar{g} \rangle_{L^2(\mathbb{T})} \\ &= 0, \end{aligned}$$

which implies that $\bar{\theta}z\mathcal{Q}_\theta \perp (\theta H^2(\mathbb{T}))^{\text{conj}}$. As a result, $\bar{\theta}z\mathcal{Q}_\theta \subseteq \mathcal{Q}_\theta^{\text{conj}}$. Finally

$$zH^2(\mathbb{T}) = z\mathcal{Q}_\theta \oplus z\theta H^2(\mathbb{T}),$$

yields

$$\begin{aligned} \bar{\theta}zH^2(\mathbb{T}) &= \bar{\theta}z\mathcal{Q}_\theta + zH^2(\mathbb{T}) \\ &\subseteq \mathcal{Q}_\theta^{\text{conj}} + zH^2(\mathbb{T}), \end{aligned}$$

and completes the proof of the lemma. \square

We are now almost ready to prove Sarason's commutant lifting theorem. Just one more result is required with regard to representations of polynomials as the sum of $H^\infty(\mathbb{D})$ -functions. Since this result holds true in several variables and is of independent interest, we prove it in the later part of this paper (see Proposition 10.7).

Theorem 9.2. *Contractive module maps on quotient modules of $H^2(\mathbb{T})$ are liftable.*

Proof. Since we are dealing with one variable quotient module, we fix a quotient module \mathcal{Q}_θ of $H^2(\mathbb{T})$ corresponding to an inner function $\theta \in H^\infty(\mathbb{D})$. Since $\mathcal{M}_1 = \{0\}$ and $H_0^2(\mathbb{T}) = zH^2(\mathbb{T})$, it follows that

$$\mathcal{M}_{\mathcal{Q}_\theta} = \mathcal{Q}_\theta^{\text{conj}} \oplus zH^2(\mathbb{T}),$$

and hence Lemma 9.1 yields a compact form of $\mathcal{M}_{\mathcal{Q}_\theta}$ as

$$\mathcal{M}_{\mathcal{Q}_\theta} = \bar{\theta}(zH^2(\mathbb{T})).$$

Let $X \in \mathcal{B}_1(\mathcal{Q})$ and suppose $\psi = X(P_{\mathcal{Q}_\theta}1)$. In view of the above and Theorem 4.1, it is enough to prove that $X_{\mathcal{Q}_\theta} : (\mathcal{M}_{\mathcal{Q}_\theta}, \|\cdot\|_1) \rightarrow \mathbb{C}$ is a contraction, where

$$X_{\mathcal{Q}_\theta}(\bar{\theta}f) = \int_{\mathbb{T}} \psi \bar{\theta}f d\mu,$$

for all $f \in zH^2(\mathbb{T})$. To this end, fix $f \in zH^2(\mathbb{T})$. Then $f \in H^2(\mathbb{T})$ and $f(0) = 0$. There exists a sequence of polynomials $\{p_m\}_{m \geq 0} \subseteq \mathbb{C}[z]$ such that

$$p_m(0) = 0,$$

for all $m \geq 0$, and

$$p_m \longrightarrow f \text{ in } H^2(\mathbb{T}).$$

Using the contractive containment $H^2(\mathbb{T}) \hookrightarrow H^1(\mathbb{T})$, we see that

$$p_m \rightarrow f \text{ in } H^1(\mathbb{T}).$$

It also follows that

$$\bar{\theta}p_m \rightarrow \bar{\theta}f, \tag{9.1}$$

in both $L^2(\mathbb{T})$ and $L^1(\mathbb{T})$. Then

$$\int_{\mathbb{T}} \psi \bar{\theta} p_m d\mu \rightarrow \int_{\mathbb{T}} \psi \bar{\theta} f d\mu,$$

and

$$\|\bar{\theta} p_m\|_1 \rightarrow \|\bar{\theta} f\|_1,$$

and hence it is enough to prove that

$$\left| \int_{\mathbb{T}} \psi \bar{\theta} p d\mu \right| \leq \|\bar{\theta} p\|_1,$$

for all $p \in \mathbb{C}[z]$ such that $p(0) = 0$. Fix such a polynomial p . Consider the inner-outer factorization of p as

$$p = \eta h,$$

where η is an inner function, h is outer, and $\eta(0) = 0$. Since $p \in H^\infty(\mathbb{D})$, it follows that $h \in H^\infty(\mathbb{D})$. Using the fact that $\sqrt{h} \in H^\infty(\mathbb{D}) \subseteq H^2(\mathbb{T})$, we rewrite p as

$$p = (\eta \sqrt{h}) \sqrt{h}.$$

It is easy to see that

$$\|p\|_1 = \|\sqrt{h}\|_2^2.$$

Moreover, we have a sequence of polynomials $\{q_t\}_{t \geq 0} \subseteq \mathbb{C}[z]$ such that

$$q_t \rightarrow \sqrt{h} \text{ in } H^2(\mathbb{T}).$$

As $\eta \sqrt{h} \in H^\infty(\mathbb{D})$, we have

$$\langle \psi, \overline{\theta q_t \eta \sqrt{h}} \rangle \rightarrow \langle \psi, \overline{\theta \sqrt{h} \eta \sqrt{h}} \rangle,$$

and then, rewriting $\sqrt{h} \eta \sqrt{h} = p$, we conclude that

$$\langle \psi, \overline{\theta q_t \eta \sqrt{h}} \rangle \rightarrow \langle \psi, \overline{\theta p} \rangle = \int_{\mathbb{T}} \psi \bar{\theta} p d\mu,$$

as $t \rightarrow \infty$. Since $(\eta \sqrt{h})(0) = 0$, Lemma 9.1 implies

$$\overline{\theta \eta \sqrt{h}} \in \mathcal{Q}_\theta \oplus \overline{z H^2(\mathbb{T})},$$

and consequently

$$\tilde{h} := P_{H^2(\mathbb{T})}(\overline{\theta \eta \sqrt{h}}) \in \mathcal{Q}_\theta.$$

Then, recalling $\psi = X(P_{\mathcal{Q}_\theta} 1)$, we compute

$$\begin{aligned} \langle \psi, \overline{\theta q_t \eta \sqrt{h}} \rangle &= \langle \psi q_t, \overline{\theta \eta \sqrt{h}} \rangle \\ &= \langle P_{H^2(\mathbb{T})} \psi q_t, P_{H^2(\mathbb{T})} \overline{\theta \eta \sqrt{h}} \rangle \\ &= \langle P_{H^2(\mathbb{T})} \psi q_t, \tilde{h} \rangle \\ &= \langle P_{\mathcal{Q}_\theta} \psi q_t, \tilde{h} \rangle. \end{aligned}$$

We also observe, for a general polynomial $r \in \mathbb{C}[z]$, that

$$\begin{aligned} X P_{\mathcal{Q}_\theta} r &= X r(S_z) P_{\mathcal{Q}_\theta} 1 \\ &= r(S_z) X P_{\mathcal{Q}_\theta} 1 \\ &= r(S_z) \psi, \end{aligned}$$

that is, $XP_{\mathcal{Q}_\theta}r = P_{\mathcal{Q}_\theta}r\psi$. It is important to note that (by virtue of Proposition 10.7)

$$P_{\mathcal{Q}_\theta}r \in H^\infty(\mathbb{D}).$$

Since $\{q_t\}_{t \geq 0} \subseteq \mathbb{C}[z]$, we conclude

$$\langle \psi, \overline{\theta q_t \eta \sqrt{h}} \rangle = \langle XP_{\mathcal{Q}_\theta}q_t, \tilde{h} \rangle,$$

and hence

$$\left| \langle XP_{\mathcal{Q}_\theta}q_t, \tilde{h} \rangle \right| \longrightarrow \left| \int_{\mathbb{T}} \psi \bar{\theta} p d\mu \right|,$$

as $t \rightarrow \infty$. But, $\|X\| \leq 1$, and $\|\tilde{h}\| \leq \|\sqrt{h}\|$, and hence

$$\left| \langle XP_{\mathcal{Q}_\theta}q_t, \tilde{h} \rangle \right| \leq \|q_t\|_2 \|\sqrt{h}\|_2.$$

As $t \rightarrow \infty$, we have (note that θ is an inner function)

$$\|q_t\|_2 \|\sqrt{h}\|_2 \rightarrow \|\sqrt{h}\|_2^2 = \|p\|_1 = \|\bar{\theta}p\|_1,$$

and hence

$$\left| \int_{\mathbb{T}} \psi \bar{\theta} p d\mu \right| \leq \|\bar{\theta}p\|_1,$$

which completes the proof of the theorem. \square

Sarason's proof of the above theorem used similar one-variable tools.

10. OTHER RESULTS

In this section, we present a variety of results with varying flavors. First, we present a solution to the Carathéodory-Fejér interpolation problem on \mathbb{D}^n . Then we discuss the interpolation problem from the standpoint of Pick matrix positivity. The lifting theorem for the Bergman space over \mathbb{D}^n is then compared, followed by decompositions of polynomials in light of Beurling-type quotient modules of $H^2(\mathbb{T}^n)$.

10.1. Carathéodory-Fejér interpolation. We use the notations that were introduced in Section 3. Recall that for $t \in \mathbb{Z}_+$, $H_t \subseteq \mathbb{C}[z_1, \dots, z_n]$ is the complex vector space of homogeneous polynomials of degree t . Moreover, for each $m \in \mathbb{N}$, define the finite-dimensional homogeneous quotient module \mathcal{Q}_m of $H^2(\mathbb{T}^n)$ by

$$\mathcal{Q}_m := \bigoplus_{t=0}^m H_t.$$

Fix a natural number m . Given $p \in \mathbb{C}[z_1, \dots, z_n]$, it follows that $p \in \mathcal{Q}_m$ if and only if $\deg p \leq m$. In the context of $\mathcal{S}(\mathbb{D}^n)$, the Carathéodory-Fejér interpolation problem asks the following: Given a polynomial $p \in \mathcal{Q}_m$, when does there exist a function $f \in \mathcal{Q}_m^\perp$ such that

$$p \oplus f \in \mathcal{S}(\mathbb{D}^n)?$$

Here and in what follows, $p \oplus f$ is in the sense of the direct sum $\mathcal{Q}_m \oplus \mathcal{Q}_m^\perp$. This formulation of the Carathéodory-Fejér interpolation problem is more appropriate for the case of $n > 1$, see [13, page 670].

The following is an interpretation of the Carathéodory-Fejér problem in terms of comutant lifting.

Proposition 10.1. *Let $p \in \mathcal{Q}_m$. There exists $f \in \mathcal{Q}_m^\perp$ such that $p \oplus f \in \mathcal{S}(\mathbb{D}^n)$ if and only if S_p is a contraction and admits a lift.*

Proof. Suppose there exists a function $f \in \mathcal{Q}_m^\perp$ such that

$$\varphi := p \oplus f \in \mathcal{S}(\mathbb{D}^n).$$

For each $q \in \mathcal{Q}_m$, we have

$$\begin{aligned} S_\varphi q &= P_{\mathcal{Q}_m} T_\varphi q \\ &= P_{\mathcal{Q}_m} (p \oplus f) q \\ &= P_{\mathcal{Q}_m} (pq) + P_{\mathcal{Q}_m} (fq). \end{aligned}$$

But, \mathcal{Q}_m^\perp is a submodule and q is a polynomial. This implies $f q \in \mathcal{Q}_m^\perp$, and consequently

$$S_\varphi q = P_{\mathcal{Q}_m} (pq).$$

On the other hand, $q \in \mathcal{Q}_m$ and

$$p \in \mathcal{Q}_m \subseteq \mathbb{C}[z_1, \dots, z_n]$$

yield

$$\begin{aligned} P_{\mathcal{Q}_m} (pq) &= P_{\mathcal{Q}_m} T_p|_{\mathcal{Q}_m} q \\ &= S_p q, \end{aligned}$$

which proves that $S_\varphi = S_p$. The contractivity of S_p also follows from the same of S_φ (recall that $\|\varphi\|_\infty \leq 1$).

For the reverse direction, suppose $S_p \in \mathcal{B}_1(\mathcal{Q}_m)$ admits a lift. Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $S_p = S_\varphi$. Using $1 \in \mathcal{Q}_m$, it follows that

$$\begin{aligned} p &= S_p 1 \\ &= S_\varphi 1 \\ &= P_{\mathcal{Q}_m} \varphi, \end{aligned}$$

and hence there exists $f \in \mathcal{Q}_m^\perp$ such that $\varphi = p \oplus f$. This completes the proof of the proposition. \square

We are now ready for the solution to the Carathéodory-Fejér interpolation problem. We will apply our commutant lifting theorem to the above. In view of Theorem 4.1, we set

$$\mathcal{M}_{\mathcal{Q}_m} = \mathcal{Q}_m^{\text{conj}} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Recall that

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{\text{conj}} \dot{+} H^2(\mathbb{T}^n)).$$

Corollary 10.2. *Given $p \in \mathcal{Q}_m$, there exists $f \in \mathcal{Q}_m^\perp$ such that $p \oplus f \in \mathcal{S}(\mathbb{D}^n)$ if and only if $\mathfrak{C}_{\mathcal{Z}, \mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1) \rightarrow \mathbb{C}$ is a contraction, where*

$$\mathfrak{C}_{\mathcal{Z}, \mathcal{W}}(g) = \int_{\mathbb{T}^n} p g d\mu \quad (g \in \mathcal{M}_{\mathcal{Q}_m}).$$

Proof. By Theorem 4.1 and the preceding proposition, the assertion is equivalent to the contractivity of the functional $\chi_{\mathcal{Q}_m}$ on $(\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1)$, where

$$\chi_{\mathcal{Q}_m} g = \int_{\mathbb{T}^n} \psi g d\mu \quad (g \in \mathcal{M}_{\mathcal{Q}_m}),$$

and $\psi = S_p(P_{\mathcal{Q}_m} 1)$. However, $1 \in \mathcal{Q}_m$ implies $P_{\mathcal{Q}_m}(1) = 1$, and $p \in \mathcal{Q}_m$ implies $S_p(1) = p$. Then

$$\chi_{\mathcal{Q}_m} = \mathfrak{C}_{\mathcal{Z}, \mathcal{W}} \text{ on } \mathcal{M}_{\mathcal{Q}_m},$$

completes the proof of the corollary. \square

We refer the reader to Eschmeier, Patton and Putinar [32], and Woerdeman [67] for the Carathéodory interpolation problem in the context of Agler-Herglotz class functions and Agler-Herglotz-Nevalinna formula on the polydisc. Also see the paper by Kalyuzhnyi-Verbovetskii [41].

10.2. Weak interpolation. Given $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$, we define the $m \times m$ Pick matrix $\mathfrak{P}_{\mathcal{Z}, \mathcal{W}}$ as

$$\mathfrak{P}_{\mathcal{Z}, \mathcal{W}} = \left((1 - w_i \bar{w}_j) \mathbb{S}(z_i, z_j) \right)_{i,j=1}^m.$$

Recall that a matrix $(a_{ij})_{m \times m}$ is positive semi-definite (in short $(a_{ij})_{m \times m} \geq 0$) if

$$\sum_{i,j=1}^m \bar{\alpha}_i \alpha_j a_{ij} \geq 0,$$

for all scalars $\{\alpha_i\}_{i=1}^m \subseteq \mathbb{C}$.

Definition 10.3. A set of distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ is said to be a Pick set if, for $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ satisfying

$$\mathfrak{P}_{\mathcal{Z}, \mathcal{W}} \geq 0,$$

there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, \dots, m$.

This definition is in view of the classical Pick positivity and the Nevanlinna-Pick interpolation on \mathbb{D} . We need another definition along the lines of Sarason's commutant lifting theorem:

Definition 10.4. A quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ satisfies the commutant lifting property if every contraction on \mathcal{Q} admits lifting.

In other words, for a module map $X \in \mathcal{B}_1(\mathcal{Q})$, there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $X = S_\varphi$. Now we use Sarason's trick to prove the Nevanlinna-Pick interpolation but in the setting of $\mathcal{S}(\mathbb{D}^n)$ for any $n \geq 1$. The proof is standard and follows in Sarason's footsteps.

Proposition 10.5. Let $\mathcal{Z} = \{z_j\}_{j=1}^m \subset \mathbb{D}^n$ be a set of m distinct points. Then \mathcal{Z} is a Pick set if and only if $\mathcal{Q}_{\mathcal{Z}}$ satisfies the commutant lifting property, where

$$\mathcal{Q}_{\mathcal{Z}} = \text{span}\{\mathbb{S}(\cdot, z_i) : i = 1, \dots, m\}.$$

Proof. We begin with a simple observation. Given $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$, we define $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by (note that $\mathcal{Q}_{\mathcal{Z}}$ is a finite-dimensional Hilbert space)

$$X\mathbb{S}(\cdot, z_i) = \bar{w}_i \mathbb{S}(\cdot, z_i) \quad (i = 1, \dots, m).$$

By Lemma 6.3, it follows that X^* is a module map. Moreover, we have

$$\left\langle (I_{\mathcal{Q}_{\mathcal{Z}}} - X^* X) \left(\sum_{j=1}^m \alpha_j \mathbb{S}(\cdot, z_j) \right), \left(\sum_{i=1}^m \alpha_i \mathbb{S}(\cdot, z_i) \right) \right\rangle = \sum_{i,j=1}^m \alpha_j \bar{\alpha}_i (1 - w_i \bar{w}_j) \mathbb{S}(z_i, z_j),$$

for all scalars $\{\alpha_i\}_{i=1}^m \subset \mathbb{C}$. It follows that X is a contraction if and only if

$$\mathfrak{P}_{\mathcal{Z}, \mathcal{W}} \geq 0.$$

Now suppose that \mathcal{Z} is a Pick set, and suppose $Y \in \mathcal{B}_1(\mathcal{Q}_{\mathcal{Z}})$ is a module map. We claim that Y has a lift. If we define $X := Y^*$, then we are precisely in the setting of the above discussion. The contractivity of X (as $\|Y^*\| \leq 1$) then implies that the Pick matrix is positive, that is, $\mathfrak{P}_{\mathcal{Z}, \mathcal{W}} \geq 0$. There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, \dots, m$. Then

$$Y^* \mathbb{S}(\cdot, z_j) = T_{\varphi}^* \mathbb{S}(\cdot, z_j) \quad (j = 1, \dots, m),$$

and we conclude that $Y^* = T_{\varphi}^*|_{\mathcal{Q}_{\mathcal{Z}}}$, or equivalently, $Y = S_{\varphi}$.

To show the converse, assume that $\mathcal{Q}_{\mathcal{Z}}$ satisfies the commutant lifting property. Suppose $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$, and let $\mathfrak{P}_{\mathcal{Z}, \mathcal{W}} \geq 0$. Then X , as defined at the beginning of the proof, is a contraction, and hence $X = S_{\varphi}$ for some $\varphi \in \mathcal{S}(\mathbb{D}^n)$. It is now routine to check that $\varphi(z_i) = w_i$ for all $i = 1, \dots, m$. \square

In the case of $n = 1$, the classical Nevanlinna Pick interpolation theorem now follows directly from Sarason's lifting theorem. The above formulation also works verbatim the same way as for multiplier spaces for general reproducing kernel Hilbert spaces over domains in \mathbb{C}^n (including the open unit ball in \mathbb{C}^n).

In view of the above proposition, we conclude that the solution to the interpolation problem in terms of Pick positivity is simply equivalent to the commutant lifting problem for quotient modules of the form $\mathcal{Q}_{\mathcal{Z}}$ for finite subsets $\mathcal{Z} \subseteq \mathbb{D}^n$. Again, this is true for general multiplier spaces.

10.3. Bergman space and lifting. Although all of the observations in this subsection hold true for weighted Bergman spaces (even for a large class of reproducing kernel Hilbert spaces) over \mathbb{D}^n along with verbatim proofs, we will stick to the Bergman space only. Denote by $A^2(\mathbb{D}^n)$ the Bergman space over \mathbb{D}^n . Recall that an analytic function f on \mathbb{D}^n is in $A^2(\mathbb{D}^n)$ if and only if

$$\|f\|_{A^2(\mathbb{D}^n)} := \left(\int_{\mathbb{D}^n} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\sigma(z)$ denotes the normalized volume measure on \mathbb{D}^n . We know that $A^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space corresponding to the Bergman kernel

$$K(z, w) = \prod_{i=1}^n \frac{1}{(1 - z_i \bar{w}_i)^2} \quad (z, w \in \mathbb{D}^n).$$

Recall that the multiplier space of $A^2(\mathbb{D}^n)$ is again $H^\infty(\mathbb{D}^n)$, which for simplicity of notation (or, to avoid confusion), we denote by $\mathcal{M}(A^2(\mathbb{D}^n))$. In other words

$$\mathcal{M}(A^2(\mathbb{D}^n)) = H^\infty(\mathbb{D}^n).$$

For each $\varphi \in \mathcal{M}(A^2(\mathbb{D}^n))$, the map $f \in A^2(\mathbb{D}^n) \mapsto \varphi f \in A^2(\mathbb{D}^n)$ defines a multiplication operator on $A^2(\mathbb{D}^n)$, which we denote by M_{φ} .

Let $\mathcal{Q} \subseteq A^2(\mathbb{D}^n)$ be a quotient module (that is, \mathcal{Q} is closed and $M_{z_i}^* \mathcal{Q} \subseteq \mathcal{Q}$ for all $i = 1, \dots, n$). For each $\varphi \in H^\infty(\mathbb{D}^n)$, set

$$B_{\varphi} = P_{\mathcal{Q}} M_{\varphi}|_{\mathcal{Q}}.$$

Suppose $X \in \mathcal{B}_1(\mathcal{Q})$ is a module map, that is, $XB_{z_i} = B_{z_i}X$ for all $i = 1, \dots, n$. We say that X is *liftable* or X has a *lift* if there exists $\varphi \in H^\infty(\mathbb{D}^n) = \mathcal{M}(A^2(\mathbb{D}^n))$ such that

$$X = B_\varphi,$$

and

$$\|B_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} \leq 1. \quad (10.1)$$

We are interested in the commutant lifting for finite-dimensional zero-based quotient modules of $A^2(\mathbb{D}^n)$. For a set of distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$, we define (following Section 6) the m -dimensional zero-based quotient module $\mathcal{B}_\mathcal{Z} \subseteq A^2(\mathbb{D}^n)$ as

$$\mathcal{B}_\mathcal{Z} = \text{span}\{K(\cdot, z_i) : i = 1, \dots, m\} \subseteq A^2(\mathbb{D}^n).$$

At the same time, keep in mind that $\mathcal{Q}_\mathcal{Z}$ is also a zero-based quotient module of $H^2(\mathbb{T}^n)$ (again, see the preceding subsection or Section 6), where

$$\mathcal{Q}_\mathcal{Z} = \text{span}\{\mathbb{S}(\cdot, z_i) : i = 1, \dots, m\} \subseteq H^2(\mathbb{T}^n).$$

Note that module maps on $\mathcal{Q}_\mathcal{Z}$ are parameterized by m scalars. To be more precise, let $X \in \mathcal{B}(\mathcal{Q}_\mathcal{Z})$. Then X is a module map if and only if there exists $\{w_i\}_{i=1}^m \subset \mathbb{C}$ such that

$$X^*\mathbb{S}(\cdot, z_i) = w_i\mathbb{S}(\cdot, z_i),$$

for all $i = 1, \dots, m$. This was observed in Lemma 6.3. The same conclusion and proof apply to $\mathcal{B}_\mathcal{Z}$. Therefore, a module map $X \in \mathcal{B}(\mathcal{Q}_\mathcal{Z})$ is associated with $\{w_i\}_{i=1}^m \subset \mathbb{C}$, which further defines a module map $\tilde{X} \in \mathcal{B}(\mathcal{B}_\mathcal{Z})$ as

$$\tilde{X}^*K(\cdot, z_i) = w_iK(\cdot, z_i),$$

for all $i = 1, \dots, m$. Consequently, we have the bijective correspondence

$$X \in \mathcal{B}(\mathcal{Q}_\mathcal{Z}) \longleftrightarrow \tilde{X} \in \mathcal{B}(\mathcal{B}_\mathcal{Z}).$$

In the case of $n = 1$, the problem of commutant lifting for quotient module $\mathcal{B}_\mathcal{Z}$ of $A^2(\mathbb{D})$ was studied in the thesis of Sultanic [65]. While she was focused on finite-dimensional quotient modules of $A^2(\mathbb{D})$, but the zero-based quotient modules played the most crucial role. Here we aim at proving the following proposition:

Proposition 10.6. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be a set of distinct points, and let $X \in \mathcal{B}_1(\mathcal{Q}_\mathcal{Z})$ be a module map. Then X on $\mathcal{Q}_\mathcal{Z}$ is liftable if and only if \tilde{X} on $\mathcal{B}_\mathcal{Z}$ is liftable.*

Proof. We start by stating a general (and well known) fact: Let $\varphi \in \mathcal{M}(A^2(\mathbb{D}^n))$. Then the operator norm (or multiplier norm) of M_φ on $A^2(\mathbb{D}^n)$ is given by

$$\|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} = \|\varphi\|_\infty.$$

Indeed, for $f \in A^2(\mathbb{D}^n)$, we have

$$\begin{aligned} \|\varphi f\|_{A^2(\mathbb{D}^n)} &= \left(\int_{\mathbb{D}^n} |\varphi f|^2 d\sigma \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{D}^n} \|\varphi\|_\infty^2 |f|^2 d\sigma \right)^{\frac{1}{2}} \\ &\leq \|\varphi\|_\infty \left(\int_{\mathbb{D}^n} |f|^2 d\sigma \right)^{\frac{1}{2}}, \end{aligned}$$

that is, $\|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} \leq \|\varphi\|_\infty$. On the other hand, for each $w \in \mathbb{D}^n$,

$$\begin{aligned}\varphi(w) &= \frac{1}{\|K(\cdot, w)\|^2} \langle K(\cdot, w), \overline{\varphi(w)} K(\cdot, w) \rangle \\ &= \frac{1}{\|K(\cdot, w)\|^2} \langle K(\cdot, w), T_\varphi^* K(\cdot, w) \rangle \\ &= \left\langle T_\varphi \left(\frac{K(\cdot, w)}{\|K(\cdot, w)\|} \right), \frac{K(\cdot, w)}{\|K(\cdot, w)\|} \right\rangle,\end{aligned}$$

implies that $|\varphi(w)| \leq \|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))}$, and completes the proof of the claim. Now, suppose that \tilde{X} on \mathcal{B}_Z is liftable, that is $\tilde{X} = B_\varphi$ for some $\varphi \in \mathcal{M}(A^2(\mathbb{D}^n)) = H^\infty(\mathbb{D}^n)$ with $\|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} \leq 1$. In view of the above observation, we have $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Suppose $\{w_i\}_{i=1}^m \subset \mathbb{C}$ be the scalars corresponding to \tilde{X} , that is

$$\tilde{X}^* K(\cdot, z_i) = w_i K(\cdot, z_i),$$

for all $i = 1, \dots, m$. This and the equality $\tilde{X} = B_\varphi$ imply that

$$\varphi(z_i) = \bar{w}_i \quad (i = 1, \dots, m),$$

and hence $X^* = S_\varphi^*$. Therefore, $X = S_\varphi$, and hence φ is a lift of X . Proof of the reverse direction is similar. \square

In other words, the lifting problem on zero-based quotient modules of $A^2(\mathbb{D}^n)$ is equivalent to the lifting problem on zero-based quotient modules of $H^2(\mathbb{T}^n)$. In the case $n = 1$, for a module map $\tilde{X} \in \mathcal{B}_1(\mathcal{B}_Z)$, if $\|X\|_{\mathcal{B}(\mathcal{Q}_Z)} \leq 1$, then \tilde{X} can be lifted (thanks to Sarason). On the other hand, if $X \in \mathcal{B}_1(\mathcal{Q}_Z)$ is a module map, then automatically $\tilde{X} \in \mathcal{B}_1(\mathcal{B}_Z)$, and hence X has a lift.

10.4. Decompositions of polynomials. In this subsection, we decompose polynomials with respect to Beurling-type quotient modules of $H^2(\mathbb{T}^n)$. This result has already been used ($n = 1$ case) to recover Sarason's commutant lifting theorem (see Theorem 9.2).

A quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ is said to be of *Beurling type* if there exists an inner function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\mathcal{Q} = (\varphi H^2(\mathbb{T}^n))^\perp.$$

Recall that all one variable quotient modules are of Beurling type [15].

Proposition 10.7. *Let $\varphi \in \mathcal{S}(\mathbb{D}^n)$ be an inner function, and let $p \in \mathbb{C}[z_1, \dots, z_n]$. Write*

$$p = f \oplus g \in \varphi H^2(\mathbb{T}^n) \oplus (\varphi H^2(\mathbb{T}^n))^\perp.$$

Then $f, g \in H^\infty(\mathbb{D}^n)$.

Proof. It is enough to prove that $f \in H^\infty(\mathbb{D}^n)$. It is also enough to consider p as a monomial. Fix $k \in \mathbb{Z}_+^n$, and suppose

$$z^k = f \oplus g \in \varphi H^2(\mathbb{T}^n) \oplus (\varphi H^2(\mathbb{T}^n))^\perp.$$

Let $l \in \mathbb{Z}_+^n$, and suppose $l_i > k_i$ for some $i = 1, \dots, n$. Since $T_z^{*l}(z^k) = 0$, it follows that

$$T_z^{*l} f = -T_z^{*l} g.$$

Since g is in the quotient module $(\varphi H^2(\mathbb{T}^n))^\perp$, we conclude that

$$T_z^{*l} f \in (\varphi H^2(\mathbb{T}^n))^\perp.$$

Now there exists $f_1 \in H^2(\mathbb{T}^n)$ such that $f = \varphi f_1$. Consequently

$$T_z^{*l} f = T_z^{*l} \varphi f_1 \in (\varphi H^2(\mathbb{T}^n))^\perp,$$

and hence

$$\langle T_z^{*l} \varphi f_1, \varphi h \rangle = 0,$$

for all $h \in H^2(\mathbb{T}^n)$. Then, $T_\varphi^* T_z^{*l} = T_z^{*l} T_\varphi^*$ and $T_\varphi^* T_\varphi = I$ yield

$$\begin{aligned} \langle T_z^{*l} f_1, h \rangle &= \langle T_z^{*l} \varphi f_1, \varphi h \rangle \\ &= 0, \end{aligned}$$

for all $h \in H^2(\mathbb{T}^n)$ and $l \in \mathbb{Z}_+^n$ such that $l_i > k_i$ for some $i = 1, \dots, n$. Therefore

$$f_1 \in \bigcap_{|l|=|k|+1} \ker T_z^{*l}.$$

and hence

$$f_1 \in \text{span}\{z^t : t \in \mathbb{Z}_+^n, |t| \leq |k| + 1\}.$$

We conclude that

$$f \in \text{span}\{z^t \varphi : t \in \mathbb{Z}_+^n, |t| \leq |k| + 1\} \subseteq H^\infty(\mathbb{D}^n).$$

This completes the proof of the proposition. \square

A similar question could be posed for other classes of functions. What about the decomposition of a rational function with respect to a Beurling decomposition, for example?

11. CONCLUDING REMARKS

We start off by commenting on the commutant lifting theorem. Let us recall Ball, Li, Timotin, and Trent's commutant lifting theorem [12, Theorem 5.1], which is only relevant for $n = 2$ case in our context.

Theorem 11.1. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^2)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then X admits a lift if and only if there exist positive operators $G_1, G_2 \in \mathcal{B}(\mathcal{Q})$ such that $G_1 - S_{z_2} G_1 S_{z_2}^* \geq 0$ and $G_2 - S_{z_1} G_2 S_{z_1}^* \geq 0$, and*

$$I - XX^* = G_1 + G_2.$$

The proof is based on Agler's transfer function realization formula for functions in $\mathcal{S}(\mathbb{D}^2)$ (which we will comment on more about below). In contrast to the preceding theorem, however, our commutant lifting theorem appears to be more explicit. For instance, Theorem 4.1 has been validated for the examples constructed in Corollary 3.3 (see Section 8).

Now we turn to the interpolation problem. We already mentioned in Section 1 that the traditional approach to solving the interpolation problem in terms of the positivity of the Pick matrix (or family of Pick matrices) in higher variables produces only limited results. There is, however, likely to be one notable exception: interpolation on \mathbb{D}^2 , which Agler [5, 6] pioneered in his seminal papers in the late '80s (also see [7, Theorem 1.3]):

Theorem 11.2. *Let $\{(\alpha_i, \beta_i)\}_{i=1}^m$ be a set of distinct points in \mathbb{D}^2 and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$. There exists $\varphi \in \mathcal{S}(\mathbb{D}^2)$ such that*

$$\varphi(\alpha_i, \beta_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if there exist positive semi-definite $m \times m$ matrices $\Gamma = (\Gamma_{ij})$ and $\Delta = (\Delta_{ij})$ such that

$$(1 - \bar{w}_i w_j) = (1 - \bar{\alpha}_i \alpha_j) \Gamma_{ij} + (1 - \bar{\beta}_i \beta_j) \Delta_{ij},$$

for all $i, j = 1, \dots, m$.

This is clearly an analogue of the solution to the classical Nevanlinna–Pick interpolation problem (also see Cole and Wermer [20, 21, 22]). In a slightly different context, see Kosiński [42] for three-point interpolation problem (also, see Cotlar and Sadosky [23, 24, 25]). Whereas the above result appears to be abstract (particularly the existence of positive semi-definite matrices), the approach is useful in a variety of other problems. Indeed, based on the Ando dilation and the von Neumann inequality for pairs of commuting contractions [8], Agler derived a realization formula for Schur functions in terms of colligation matrices, which leads to the above solution to the interpolation problem. His realization formula has proven very useful in operator theory and function theory on \mathbb{D}^n , $n \geq 2$. Whereas we believe Theorem 6.6 is more concrete and provides a new perspective on the interpolation problem in general, we are unsure how to relate it to Theorem 11.2. We are also unclear about using Theorem 11.2 to validate the examples of interpolation in Theorem 7.4 for the specific case of $n = 2$.

Finally, we remark that, unlike the present case of scalar functions, the earlier lifting theorem and the solutions to the interpolation problem work equally well for the operator or vector-valued functions [7, 12, 11]. The powerful n -variables von Neumann inequality (which is automatic in the case of $n = 2$ but not so when $n > 2$), like the Sz.-Nagy and Foiaş [44] effective dilation theoretic approach appears to be a key factor. However, as previously stated, we followed a function theoretic route pioneered by Sarason in his work [60]. The results reported here, we think, will be also helpful in building related theories like isometric dilations for commuting contractions, several variables von Neumann inequality, Nehari problem on \mathbb{D}^n , etc., similar to Sarason’s classic result.

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